ALGEBRA PHD PRELIMINARY EXAM, 17 MAY 2018

Instructions: Put all work you wish to have graded on the blank sheets of paper provided. Use a fresh sheet of paper for each new problem, and clearly indicate on each page which problem you are answering. No books or calculators are allowed. There are 8 questions worth 10 points each, for a total of 80 points.

1. Let \( V \) be a vector space over the real numbers. Let \( U \) and \( W \) be subspaces of \( V \). Prove that \( U \cup W \) is a subspace of \( V \) if and only if either \( U \subseteq W \) or \( W \subseteq U \).

2. Let \( G \) and \( H \) be groups and consider the product group \( G \times H \). Let \( e_G \) be the identity element of \( G \). Consider the set \( X \subseteq G \times H \) defined by \( X = \{ (e_G, h) | h \in H \} \). Construct a bijective correspondence between \{subgroups of \( G \)\} and \{subgroups of \( G \times H \) that contain \( X \)\}. Be sure to prove that your bijection works.

3. Prove that there is no simple group of order 56.

4. Let \( A \) and \( B \) be \( n \times n \) matrices over the complex numbers.
   (a) Prove that if \( A \) is similar to \( B \), then \( A \) and \( B \) have the same characteristic polynomial.
   (b) Prove that if \( A \) and \( B \) are both diagonalizable and \( A \) and \( B \) have the same characteristic polynomial, then \( A \) is similar to \( B \).
   (c) Show by example that if at least one of \( A \) or \( B \) is not diagonalizable, then it can be the case that \( A \) and \( B \) have the same characteristic polynomial but \( A \) is not similar to \( B \). Be sure to prove your example is valid.

5. Let \( G \) be the Abelian group generated by four elements \( w, x, y, z \), subject to the relations

\[
\begin{align*}
y + 3z &= 0 \\
-2w + x + y + 3z &= 0 \\
-2w + 4x + y + 3z &= 0 \\
-3x + y + 5z &= 0.
\end{align*}
\]

Write \( G \) as a direct sum of cyclic groups in two ways, corresponding to the two versions of the Fundamental Theorem of Finitely Generated Abelian Groups.

6. Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Recall that \( M \) is said to be finitely generated if there are elements \( x_1, \ldots, x_n \in M \) such that \( M = Rx_1 + \cdots + Rx_n \).
(a) If $N \subseteq M$ is a submodule such that both $N$ and $M/N$ are finitely generated, prove that $M$ is finitely generated.
(b) Give an example, with justification, of a finitely generated module $M$ and a submodule $N$ which is not finitely generated.

7. Suppose that $A$ is a square complex matrix with characteristic polynomial $c_A(x) = (x-1)^4(x+3)^5$. Assume also that $A - I$ has nullity 4 and $A + 3I$ has nullity 1, where $I$ is the identity matrix of the same size as $A$. Find, with justification, all possible Jordan canonical forms of $A$, and give the minimal polynomial for each.

8. Set $K = \mathbb{Q}(i, \sqrt[4]{2})$, where $i$ is the complex root of $-1$ and $\sqrt[4]{2}$ is the real fourth root of 2.
   (a) Find the degree $[K : \mathbb{Q}]$.
   (b) Identify all the elements of $\text{Aut}_\mathbb{Q}(K)$.
   (c) Identify the isomorphism type of the group $\text{Aut}_\mathbb{Q}(K)$.
   Justify all your conclusions.