1. (10 points) Find a general solution of the differential equation $x^2y' = (1 + y^2)^{1/2}$. (You may use the formula: $\int \frac{dy}{(1+y^2)^{1/2}} = \ln |y + (1 + y^2)^{1/2}| + C$.)
2. (10 points) Find an integrating factor for the equation

\[(3xy + y^2 + 1)dx + (x^2 + xy)dy = 0\]

and then solve the equation.
3. Find general solutions for the following inhomogeneous differential equations:
(1) (8 points) $y'' + y' = e^{-t}$
(2) (8 points) $y'' + y' + y = \cos t$
(3) (9 points) \( y'' - 2y' + y = e^t \)
4. (10 points) Use the method of \textit{variation of parameters} to find the general solution of the given equation \( y'' - y' - 2y = 2e^{-t} \) and find a solution which satisfies additional conditions \( y(0) = 0 \) and \( y'(0) = 1 \). (Students who do not use the specified method will receive no credit for this problem.)
5. (10 points) Use the definition of the Laplace transform to find $\mathcal{L}(e^{-t})$. 
6. Find the inverse Laplace transform of following functions:

(1) (5 points) \( Y(s) := \frac{1}{s(s^2+1)} \)

(2) (5 points) \( Y(s) := e^{-s} \frac{1}{s(s^2+1)} \).
7. (10 points) Find the solution of the initial value problem $y'' + y = u_1(t), y(0) = 0, y'(0) = 0.
(You may make use of the result of Problem 6.)
8. (10 points) Find the general solution of the system

\[ x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x. \]
9. Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

(1) (6 points) Find an invertible matrix \( T \) that diagonalizes the matrix \( A \).

(2) (6 points) Find \( T^{-1} \) by using the row elementary operations and compute \( T^{-1}AT \).
(3) (6 points) Find the general solution of the system \( x' = Ax \) by using the result in (2).

(4) (7 points) Find the fundamental matrix \( \Phi \) associated with the initial vectors \( e^{(1)} \) and \( e^{(2)} \).
10. (20 points) Find the general solution of the system $x' = Ax + b$, where

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} e^t \\ t \end{pmatrix}.$$
MAT 414 Final Exam

2 May 2013

This exam has 9 problems on 10 pages (including this cover page). There is also a table of Laplace transforms. You have two hours to work. You must show all your work; correct answers without complete justification will not receive full credit. Calculators are allowed. Good luck!

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</table>
1. (12 pts) Solve the initial value problem and determine the largest interval in which the solution exists. Describe the behavior of the solution as $t \to \infty$.

$ty' + 2y = 4t - 2 \quad y(-3) = 2$

largest interval

behavior as $t \to \infty$
2. (12 pts) Find an equation implicitly defining the solution of the initial value problem.

\[(x + y) \, dx + (x + 2y) \, dy = 0, \quad y(4) = 1\]
3. (12 pts) Find an explicit solution (i.e. of the form $y = a$ function of $x$) of the following initial value problem.

$$y' + y^2 \sin x = 0 \quad y(0) = \frac{1}{6}$$
4. (a) (6 pts) A large tank initially holds 200 liters of water that contains 15 kg of dissolved salt. A brine (salt) solution is flowing into the tank at the rate of 5 L/min while the well mixed solution flows out of the tank at the rate of 3 L/min. The brine solution entering the tank has a salt concentration of 4 kg/L. Find an initial value problem that models the amount of salt, $Q$, in the tank at time $t$, assuming it does not overflow. DO NOT solve the IVP.
Suppose the tank holds 300 liters. When does the tank first begin to overflow?

(b) (6 pts) A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm$^3$ of the drug enters the patient’s bloodstream at a rate of 100 cm$^3$/h. The drug is absorbed by the body tissues or otherwise leaves the bloodstream at a rate proportional to the amount of the drug present with a rate constant of 0.4. Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time. DO NOT solve the differential equation.
How much of the drug is present in the bloodstream in the long run?
5. (10 pts) Find the solution of the initial value problem.

\[ y'' - 5y' - 14y = 0, \quad y(0) = 1, \quad y'(0) = -20 \]

and describe the behavior of the solution as \( t \) increases.
6. (a) (6 pts) Find the general solution of the differential equation.

\[ y'' + 2y' + 5y = 0 \]

(b) (6 pts) Find the general solution of the differential equation.

\[ y'' + 2y' + 5y = 4e^{3t} \]
7. (10 pts) The function \( y_1(t) = t \) is a solution of the differential equation

\[ t^2 y'' + 3ty' - 3y = 0, \quad t > 0. \]

Find a second solution using reduction of order. Find the general solution.
8. (a) (5 pts) Find the Laplace transform of \( f(t) = e^{-2t} \sin 3t + u_5(t)[(t - 5)^2 + 1] \).

(b) (5 pts) Find the inverse Laplace transform of \( F(s) = \frac{e^{-2s}(5s + 6)}{s^2 + 9} \).
9. (10 pts) Find the solution of the initial value problem

\[ y'' - 2y' + 5y = \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 1 \]
MAT 414 Final Exam

2 May 2013

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Score: 100
1. (12 pts) Solve the initial value problem and determine the largest interval in which the solution exists. Describe the behavior of the solution as $t \to \infty$.

$$ty' + 2y = 4t - 2 \quad y(-3) = 2$$

largest interval ____________

behavior as $t \to \infty$ ____________
2. (12 pts) Find an equation implicitly defining the solution of the initial value problem.

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5. (10 pts) Find the solution of the initial value problem.

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and describe the behavior of the solution as \( t \) increases.
6. (a) (6 pts) Find the general solution of the differential equation.

\[ y'' + 2y' + 5y = 0 \]

(b) (6 pts) Find the general solution of the differential equation.

\[ y'' + 2y' + 5y = 4e^{3t} \]
7. (10 pts) The function \( y_1(t) = t \) is a solution of the differential equation

\[
t^2y'' + 3ty' - 3y = 0, \quad t > 0.
\]

Find a second solution using reduction of order. Find the general solution.
8. (a) (5 pts) Find the Laplace transform of \( f(t) = e^{-2t} \sin 3t + u_5(t)(t - 5)^2 + 1 \).

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9. (10 pts) Find the solution of the initial value problem

\[ y'' - 2y' + 5y = \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 1 \]
### TABLE 6.2.1 Elementary Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t) = \mathcal{L}^{-1}{F(s)}$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
<th>Notes</th>
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<tbody>
<tr>
<td>1. $1$</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
<td>Sec. 6.1; Ex. 4</td>
</tr>
<tr>
<td>2. $e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
<td>Sec. 6.1; Ex. 5</td>
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<tr>
<td>3. $t^n, n = \text{positive integer}$</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
<td>Sec. 6.1; Prob. 27</td>
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<tr>
<td>4. $p^t, p &gt; -1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}$, $s &gt; 0$</td>
<td>Sec. 6.1; Prob. 27</td>
</tr>
<tr>
<td>5. $\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$, $s &gt; 0$</td>
<td>Sec. 6.1; Ex. 7</td>
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<tr>
<td>6. $\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
<td>Sec. 6.1; Prob. 6</td>
</tr>
<tr>
<td>7. $\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>8. $\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>9. $e^{at}\sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$, $s &gt; a$</td>
<td>Sec. 6.1; Prob. 13</td>
</tr>
<tr>
<td>10. $e^{at}\cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$, $s &gt; a$</td>
<td>Sec. 6.1; Prob. 14</td>
</tr>
<tr>
<td>11. $t^n e^{at}, n = \text{positive integer}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$, $s &gt; a$</td>
<td>Sec. 6.1; Prob. 18</td>
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<tr>
<td>12. $u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$, $s &gt; 0$</td>
<td>Sec. 6.3</td>
</tr>
<tr>
<td>13. $u_c(t)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
<td>Sec. 6.3</td>
</tr>
<tr>
<td>14. $e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
<td>Sec. 6.3</td>
</tr>
<tr>
<td>15. $f(ct)$</td>
<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c &gt; 0$</td>
<td>Sec. 6.3; Prob. 25</td>
</tr>
<tr>
<td>16. $\int_0^t f(t-\tau)g(\tau),d\tau$</td>
<td>$F(s)G(s)$</td>
<td>Sec. 6.6</td>
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<tr>
<td>17. $\delta(t-c)$</td>
<td>$e^{-cs}$</td>
<td>Sec. 6.5</td>
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<tr>
<td>18. $f^{(n)}(t)$</td>
<td>$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$</td>
<td>Sec. 6.2</td>
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<tr>
<td>19. $(-t)^nf(t)$</td>
<td>$F^{(n)}(s)$</td>
<td>Sec. 6.2; Prob. 28</td>
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Laplace transforms:

<table>
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<th>$f(t)$</th>
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<tr>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
</tr>
<tr>
<td>$\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
</tr>
<tr>
<td>$\delta_c$</td>
<td>$e^{-cs}$</td>
</tr>
</tbody>
</table>
1. Solve the differential equation \( y' = 6t^2 e^{2y} - 12 e^{2y} \) with the initial condition \( y(1) = 0 \).
2. Find the general solution of the equation $y'' + 8y' + 20y = 0$. 
3. For the autonomous equation \( y' = 25y^2 - y^3 \), sketch the phase line, find all equilibria and determine the stability of each equilibrium.
4. Find a particular solution of the equation \( y'' - 4y' + 4y = 6t \).
5. Given the piecewise defined function

\[ g(t) = \begin{cases} 
3 - 2t, & t < 2 \\
2t - 8, & 2 \leq t < 8 \\
0, & t \geq 8 
\end{cases} \]

(a) Write a formula for \( g \) using the Heaviside (step) function.
(b) Find the Laplace transform of \( g \).
6. Use the Laplace transform to solve \( y'' - y' - 2y = 3 \delta_2(t) \) with initial conditions \( y(0) = 0 \) and \( y'(0) = 6 \).
7. Find the solution of the system with given initial conditions.
\[
x' = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} x \\
\bar{x}(0) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}
\]
8. Consider the linear system

\[ \dot{x} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} x \]

(a) What is the type of equilibrium at (0, 0)?
(b) Is the equilibrium asymptotically stable, stable (but not asymptotically), or unstable?
9. Consider the nonlinear system
\[
\begin{align*}
x' &= 2y \\
y' &= 6y^2 + 18xy
\end{align*}
\]
Find the equation of the trajectory that passes through the point \((0, 1)\).
10. Consider the nonlinear system

\[
\begin{aligned}
x' &= x + y^2 \\
y' &= x + 9
\end{aligned}
\]

(a) Find the equilibria of this system.
(b) For each equilibrium, determine its type and stability using the Jacobian matrix.