Math 397 Final Exam Version 1 May 5, 2014

Instructor's Name __________________________
Notes, calculators, head phones, electronic devices are not permitted. Turn off your cell phone.
There are two versions of this exam. All answers require justification.
Do not write below this line.

<table>
<thead>
<tr>
<th>Prob</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pts</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MaxPt</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Total ________________
1. Let \( \vec{a} = \langle 1, 0, 1 \rangle, \quad \vec{b} = \langle 2, 1, 0 \rangle \).

(a) Calculate \( \vec{a} \cdot \vec{b} \).

(b) Find the tangent of the angle \( \theta \) between \( \vec{a} \) between \( \vec{b} \).
2. A ball that is tied to the origin by a string 5 units long moves counter-clockwise on a circle of radius 5. The circle of motion lies in the plane $6x - 8y + 2z = 0$. At the instant the ball reaches the point $(4, 3, 0)$ the string breaks and the ball flies off on a tangent. Find parametric equations for the subsequent line of motion. (Assume the ball is subject to no external forces after the string breaks.)
3. Determine if the following limits exist. If the limit exists, find the limit; if it does not exist, show that it does not exist. Justify your answers.

\[
\begin{align*}
(a) \quad & \lim_{(x,y) \to (0,0)} \frac{x^2 \cos \left( \frac{\pi y}{2} \right)}{x^2 + y^2} \\
(b) \quad & \lim_{(x,y) \to (0,1)} \frac{(y-1)^2 x^3}{(y-1)^2 + x^7}
\end{align*}
\]
4. Given \( z = x^2 - xy + 5e^y \), where \( x = \cos(s+t) \) and \( y = \sin(s-t) \), find \( \frac{\partial z}{\partial t} \). (Note: express your answer in terms of the variables \( s, t \).)
5. Let $S$ be the surface given by the equation $x^4 + y^4 + z^4 = 3x^2y^2z^2$ and let $P$ be the point $(1, 1, 1)$.

(a) Find the equation of the tangent plane to $S$ at $P$.

(b) Find the equation of the normal line to $S$ at $P$. 
6. Given the function $f(x, y) = x^4 + y^4 - xy$, find all critical points and classify them.
7. Evaluate the following iterated integral.

\[ \int_{2}^{4} \int_{0}^{x} \frac{x}{e^{y}} \, dy \, dx \]
8. Consider the solid in the first octant bounded by the coordinate planes, the cylinder \( x^2 + y^2 = 2 \), and by \( z = x^2 + y^2 \). Suppose the material in the solid has mass density function \( \rho(x, y, z) = xyz \). Set up, but do NOT evaluate, the triple integral in cylindrical coordinates needed to find the total mass of the solid.
9. Using spherical coordinates find the volume of the solid that lies within the sphere
\( x^2 + y^2 + z^2 = 4 \), above the \( xy \)-plane, and below the cone \( z = \sqrt{x^2 + y^2} \).
10. Consider the vector field \( \vec{F}(x, y) = y \vec{i} - x \vec{j} \)

(a) Calculate the integral \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the circle of radius 1 in the x-y plane centered at the origin.

(b) Show that \( \vec{F}(x, y) \) is not a conservative vector field.
11. Use Green’s Theorem to evaluate the given line integral along the positively oriented (i.e. counter-clockwise) curve $C$; $x^2 + y^2 = 4$.
\[
\int_C y^3 \, dx - x^3 \, dy
\]
1. Let $\vec{a} = \langle 1, 2, 0 \rangle$, $\vec{b} = \langle 2, 2, 1 \rangle$.

(a) Calculate $\vec{a} \cdot \vec{b}$.

(b) Find the cotangent of the angle $\theta$ between $\vec{a}$ between $\vec{b}$. 

2. A ball that is tied to the origin by a string 5 units long moves counter-clockwise on a
circle of radius 5. The circle of motion lies in the plane $3x - 4y + 2z = 0$. At the instant the
ball reaches the point $(4, 3, 0)$ the string breaks and the ball flies off on a tangent. Find
parametric equations for the subsequent line of motion. (Assume the ball is subject to no
external forces after the string breaks.)
3. Determine if the following limits exist. If the limit exists, find the limit; if it does not exist, show that it does not exist. Justify your answers.

(a) \[ \lim_{{(x,y) \to (0,0)}} \frac{x^3 \cos \left( \frac{\pi y}{2} \right)}{x^2 + y^2} \]

(b) \[ \lim_{{(x,y) \to (0,1)}} \frac{(y-1)^2 x^3}{(y-1)^2 + x^7} \]
4. Given $z = 2x^2 - xy + 5e^x$, where $x = \cos(s + t)$ and $y = \sin(s - t)$, find $\frac{\partial z}{\partial s}$. (Note: express your answer in terms of the variables $s, t$. )
5. Let $S$ be the surface given by the equation $2x^4 + 2y^4 + 2z^4 = 3x^2y^2 + 3x^2z^2$ and let $P$ be the point $(1, 1, 1)$.
(a) Find the equation of the tangent plane to $S$ at $P$.

(b) Find the equation of the normal line to $S$ at $P$. 
6. Given the function \( f(x, y) = x^4 + y^4 - xy \), find all critical points and classify them.
7. Evaluate the following iterated integral.

\[ \int_0^2 \int_{2z^2}^{8} x \, dy \, dx \]
8. Consider the solid in the first octant bounded by the coordinate planes, the cylinder \( x^2 + y^2 = 2 \), and by \( z = x^2 + y^2 \). Suppose the material in the solid has mass density function \( \rho(x, y, z) = xyz \). Set up, but do NOT evaluate, the triple integral in cylindrical coordinates needed to find the total mass of the solid.
9. Using spherical coordinates find the volume of the solid that lies within the sphere \( x^2 + y^2 + z^2 = 4 \), and above the cone \( z = \sqrt{x^2 + y^2} \).
10. Consider the vector field \( \vec{F}(x, y) = yi - xj \)
(a) Calculate the integral \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the circle of radius 2 in the x-y plane centered at the origin.

(b) Show that \( \vec{F}(x, y) \) is not a conservative vector field.
11. Use Green’s Theorem to evaluate the given line integral along the positively oriented (i.e. counter-clockwise) curve \( C: \ x^2 + y^2 = 9. \)
\[
\int_C 2y^3 \, dx - 2x^3 \, dy
\]