1. Let $X$ be a metric space. Consider a family of subsets of $X$, denoted $\{E_i : i \in A\}$ where $A$ is an uncountable index set. Suppose that for every finite or countable set $B \subset A$ the intersection

$$\bigcap_{i \in B} E_i$$

is open. Prove that the set

$$E = \bigcap_{i \in A} E_i$$

is also open.

2. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that for every compact set $K \subset \mathbb{R}$ the inverse image $f^{-1}(K)$ is also compact. Prove that

$$\lim_{x \to +\infty} |f(x)| = +\infty$$

3. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has derivatives of all orders and satisfies $f(0) = f'(0) = f''(0) = 0$. Prove that the function $g(x) = f(x)^{1/3}$ is differentiable at $0$.

4. Let $f$ and $g$ be Riemann-Stieltjes integrable on $[a, b]$ with respect to a non-decreasing function $\alpha$. Suppose that given any partition $P$ of $[a, b]$ there exists a partition $Q$ of $[a, b]$ such that

$$L(f, P, \alpha) \leq L(g, Q, \alpha) \quad \text{and} \quad L(g, P, \alpha) \leq L(f, Q, \alpha)$$

Prove that

$$\int_a^b f \, d\alpha = \int_a^b g \, d\alpha$$

5. Determine all positive continuous functions $f$ on $[1, \infty)$ such that

$$\ln \left(1 + \int_0^\theta f(e^x) \, dx \right) = \theta$$

for all real numbers $\theta > 0$.

6. Prove that the image of any open set containing the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ under the mapping $f(x, y) = (x^4 + y^4, 2xy)$ is not a subset of the unit disk.