No books notes or calculators are allowed. Do all parts of all questions. There are four questions each worth 10 points. Show the work you do to obtain an answer. Give reasons for your answers.

1. Let $R$ be a ring and let

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

be a short exact sequence of left $R$-modules. Prove that the following three conditions are equivalent. Provide a proof, do not just say that these are the three equivalent conditions for a sequence to split.

(a) There exists a left $R$-module homomorphism $i : C \to B$ such that $g \circ i =$ identity.

(b) There exists a left $R$-module homomorphism $j : B \to A$ such that $j \circ f =$ identity.

(c) There exists a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & C & \longrightarrow & 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow & \\
0 & \longrightarrow & A & \overset{u}{\longrightarrow} & A \oplus C & \overset{v}{\longrightarrow} & C & \longrightarrow & 0 \\
\end{array}
$$

where all maps are left $R$-module homomorphisms, both rows are exact, $\alpha$ and $\gamma$ are identity maps, $\beta$ is a left $R$-module isomorphism, $u(a) = (a, 0)$, $v(a, c) = c$, $\beta \circ f = u \circ \alpha$, and $\gamma \circ g = v \circ \beta$.

2. Let $R$ be an integral domain and let $M$ be an $R$-module.

(a) Give the definition of $M$ being a divisible $R$-module.

(b) Prove that if $M$ is an injective $R$-module then $M$ is a divisible $R$-module.

3. Let $A$ be the ring of $3 \times 3$ invertible matrices with entries in the quaternions. Let $B$ be the ring of $2 \times 2$ invertible matrices with entries in the field with three elements. Let $V$ be the additive group of $3 \times 4$ matrices with entries in the quaternions. Matrix multiplication makes $V$ into a left $A$-module. Let
$W$ be the additive group of $2 \times 3$ matrices with entries in the field with three elements. Matrix multiplication makes $W$ into a left $B$-module. Finally $V \times W$ becomes a left $A \times B$-module by operating componentwise. For $a \in A$, $b \in B$, $v \in V$, and $w \in W$, $(a, b)(v, w) = (av, bw)$.

Prove that $V \times W$ is a semisimple left $A \times B$-module.

4. Let $k$ be a field and $k[x]$ the polynomial ring over $k$. Consider the following two multiplicatively closed subsets of $k[x]$.

$$S = \{1, x, x^2, x^3, \ldots \} = \{x^n : n \text{ is a non-negative integer}\}.$$

$$T = \{1, x^2, x^4, x^6, \ldots \} = \{x^{2n} : n \text{ is a non-negative integer}\}.$$

Prove whether or not the following two localized rings are isomorphic $S^{-1}k[x]$ and $T^{-1}k[x]$. Check everything that needs to be checked.
In what follows, $R$ is an associative ring with identity.

1. Let $M$ be a left $R$-module.
   (a) (3 points) Give the definition of the functor $\text{Hom}_R(M, -)$ being:
   (i) left exact;
   (ii) exact.
   (b) (2 points) Give the definition of $M$ being projective.
   (c) (5 points) Using that $\text{Hom}_R(M, -)$ is left exact, prove that $M$ is projective if and only if $\text{Hom}_R(M, -)$ is exact.

2. Consider the diagram of $R$-modules
   
   \[
   \begin{array}{ccc}
   A & \xrightarrow{f} & B \\
   g \downarrow & & \downarrow \\
   C & \xrightarrow{\beta} & D
   \end{array}
   \]
   (a) (3 points) Say what it means for the diagram to be a pushout of the diagram above.
   (b) (5 points) By using a suitable quotient module of $B \oplus C$ or otherwise, show that a pushout always exists.
   (c) (2 points) In what sense is the pushout of the original diagram unique?

3. In the following commutative diagram of $R$-modules the columns and the second and third rows are exact. Prove that the first row must be exact.
   This statement is known as the $3 \times 3$ Lemma. Your assignment is to provide a proof.
4. A cochain complex

\[ K^\bullet : \cdots \to K^n \xrightarrow{d^n} K^{n+1} \xrightarrow{d^{n+1}} \cdots \]

is bounded below if \( K^j = 0 \) for all \( j < i \), for some integer \( i \). Denote by \( \text{Kom} R \) the category of cochain complexes of left \( R \)-modules and denote by \( \text{Kom}^+ R \) the full subcategory of \( \text{Kom} R \) determined by the bounded below complexes.

Let \( r : Y^\bullet \to Z^\bullet \) be a quasi-isomorphism in \( \text{Kom} R \) with \( Y^\bullet \in \text{Kom}^+ R \). Prove that there is a quasi-isomorphism \( q : Z^\bullet \to U^\bullet \) in \( \text{Kom} R \) with \( U^\bullet \in \text{Kom}^+ R \). \textit{Hint:} if \( Y^j = 0 \) for all \( j < i \), set \( U^i = Z^i / \text{Im} d_{Z}^{i-1} \).