Qualifying Exam, Complex Analysis, January 2017

Directions: Attempt as many as you can of the following problems. Write neatly on one-sided sheets; explain; show work; justify your claims (if you are using a theorem from class and/or the textbook, you must quote the theorem by its name, but you are not required to supply the theorem’s proof). Write page numbers and remember to print your name.

Notation: Throughout the exam $\Delta$ denotes the open unit disc in $\mathbb{C}$ with center at the origin, that is: $\Delta = \{z \in \mathbb{C}, |z| < 1\}$.

1. Let $C$ denote the positively-oriented boundary of the domain
   
   $D = \left\{ z \in \mathbb{C}, -2 < \text{Re} \, z < \frac{1}{2}, |\text{Im} \, z| < 2 \right\}$.

   Find
   
   $I = \int_C \frac{z^n}{z^4 - 1} \, dz$

   where $n \geq 0$ is an integer. Write your answer in algebraic form $"I = a + ib"$.

2. Find the domain of convergence and the sum of the following two power series. Explain.

   (a.) $\sum_{k=1}^{\infty} k \cdot z^k$; (b.) $\sum_{k=1}^{\infty} k^2 \cdot z^k$

3. Evaluate the following integral. Explain and justify all your claims.

   $I = \int_{-\infty}^{+\infty} \frac{x^3 \sin x}{(x^2 + 1)^2} \, dx$

4. Prove that there are no, non-constant polynomials of the form

   \begin{equation}
   p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0
   \end{equation}

   that satisfy

   \begin{equation}
   |p(z)| < 1 \quad \text{when} \quad |z| = 1.
   \end{equation}