1. Consider the following proposition: Every bounded continuous real-valued function $f$ on $\mathbb{R}$ attains its maximum. The following argument which attempts to prove this has an error. (a) Find where the error occurs and (b) provide a counterexample, with details, to show that the argument indeed fails at that point:

Let $M = \sup \{ f(x) : x \in \mathbb{R} \}$, and let $x^*, x_n \in \mathbb{R}$ such that $x_n \rightarrow x^*$ and $f(x_n) \rightarrow M$. Since $f$ is continuous, $f(x_n) \rightarrow f(x^*)$, which implies $f(x^*) = M$. Hence, $x^*$ is where $f$ attains its maximum.

2. Prove: there exists $c > 0$ and continuous functions $f, g$ on $(-c, c)$ such that $f(0) = g(0) = 0$ and

$$
\sin(f(z)) + \cos(g(z)) = z^2 + 1, \quad \text{and} \quad (f(z))^2 + 2e^{2g(z)} = 2 \cos z
$$

for all $z \in (-c, c)$.

3. Let $f$ be continuously differentiable, and suppose that $f(0) < -1$, $f(1) > 0$, and $f(2) < 0$. Prove that for each $c \in [0, 1]$ there exists $x_c \in (0, 2)$ such that $f'(x_c) = c$.

4. Let $(X, d)$ be a metric space. Prove or provide a counterexample:
   
   (a) The intersection of finitely many dense subsets of $X$ is dense.
   
   (b) The intersection of finitely many open dense subsets of $X$ is open and dense.

5. Let $f, g$ be continuous functions on $\mathbb{R}$ such that $f$ is differentiable everywhere and let $f(1) = 0$. Prove that $fg$ is differentiable at 1.

6. Let $(f_n)$ be a sequence of functions on $[0, 1]$ with continuous first and second derivatives, such that for all $n \geq 1$,

$$
1 \leq f_n(0) \leq 2, \quad 3 \leq f_n'(0) \leq 4, \quad \sup_{0 \leq x \leq 1} |f_n''(x)| \leq 12
$$

Prove that $(f_n)$ has a subsequence which converges uniformly on $[0, 1]$. 
