Qualifying Exam, Complex Analysis, August 2016

Directions: Attempt as many as you can of the following problems. Write neatly on one-sided sheets; explain; show work; justify your claims (if you are using a theorem from class and/or the textbook, you must quote the theorem by its name, but you are not required to supply the theorem’s proof). Write page numbers and remember to print your name.

Notation: Throughout the exam $\Delta$ denotes the open unit disc in $\mathbb{C}$ with center at the origin, that is: $\Delta = \{z, |z| < 1\}$.

1. Let $C$ denote the positively-oriented boundary of the domain

$$D = \{z \in \mathbb{C}, -2 < \Re z < \frac{1}{2}, |\Im z| < 2\}.$$ 

Find

$$I = \int_{C} \frac{z^n}{z^4 - 1} \, dz$$

where $n \geq 0$ is an integer. Write your answer in algebraic form “$I = a + ib$”.

2. Let $f$ be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle $\{|z| = 1\}$. Suppose that there is an entire function $g$ such that $f(z) = g(z)$ for $|z| = 1$. Prove that $f = g$ (and hence $f$ is entire).

3. Let $S$ be a square with center at the origin. Suppose that $F : \Delta \rightarrow S$ is analytic, one-to-one and onto and furthermore, that $F(0) = 0$. Show that

$$F(iz) = iF(z) \quad \text{for all } z \in \Delta.$$ 

4. Prove that there are no, non-constant polynomials of the form

$$(0.1) \quad p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

that satisfy

$$(0.2) \quad |p(z)| < 1 \quad \text{when} \quad |z| = 1.$$