MAT 485    Final exam
May 8th, 2012

Dr. Simon Smith
Time: 2 hours

Instructions

• Show and explain all computation. You will receive no credit for answers showing no working.

• Do not round numbers during your computation, although you may give your final answer to two decimal places.

• You may use the backs of these pages for rough working, but please note that work written here will not be graded.

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<th>Problem</th>
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Question 1. [10pts] Solve the initial value problem,

\[ y' = -\frac{1}{t} y + t^2 + 2, \quad y(1) = 2. \]
Question 2. [10pts] Let

\[ A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}. \]

(a) Find the determinant of the matrix \( A \)

(b) Find the inverse of matrix \( A \), if it exists.
Question 3. [10pts] Find the general solution to the differential equation,

\[ y'' + 2y' + 2y = 0. \]
Question 4. [10pts] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ 0 \\ z \end{pmatrix}.$$ 

(a) Find standard matrix of $T$.

(b) Find the kernel of $T$. 
Question 5. [10pts] Consider the following system of differential equations,

\begin{align*}
x' &= x + 2y \\
y' &= 2x + y
\end{align*}

(a) Find general solution to the above system.

(b) The above system has an equilibrium solution at \( x = 0, y = 0 \). Classify the type (attracting node, repelling node or saddle) of the solution, and say whether or not it is stable or unstable. Circle the words that apply.

<table>
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<tr>
<th>Attracting node</th>
<th>Repelling node</th>
<th>Saddle</th>
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<td>Stable</td>
<td>Unstable</td>
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Question 6. [10pts]
Find the general solution to the following differential equation,

\[ y'' + 4y' + 3y = t^2 + 1. \]
**Question 7.** [10pts] Find the general solution to the initial value problem,

\[ y'' + y' - 2y = H(t - 2), \quad y(0) = 0, \quad y'(0) = 0, \]

where \( H(t) \) denotes the *Heaviside function* or *Step function*. Note that the Laplace transforms of common functions are listed on the back page of this exam.
Laplace transform formulae:

Let \( F(s) = \mathcal{L}\{f(t)\} \).

- \( f(t) = 1; \quad F(s) = \frac{1}{s}, \quad s > 0 \)
- \( f(t) = t^n; \quad F(s) = \frac{n!}{s^{n+1}}, \quad s > 0, \quad n \) a positive integer
- \( f(t) = e^{at}; \quad F(s) = \frac{1}{s-a}, \quad s > a \)
- \( f(t) = \sin(bt); \quad F(s) = \frac{b}{s^2 + b^2}, \quad s > 0 \)
- \( f(t) = \cos(bt); \quad F(s) = \frac{s}{s^2 + b^2}, \quad s > 0 \)
- \( \mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a \)
- \( f(t) = \delta(t-a); \quad F(s) = e^{-as}, \quad s > 0, a > 0 \)
- \( f(t) = H(t-a); \quad F(s) = \frac{e^{-as}}{s}, \quad a > 0, s > 0 \), where \( H(t) \) is the Heaviside or Step function
- \( f(t) = H(t-a)g(t-a); \quad F(s) = e^{-as}\mathcal{L}\{g(t)\}, \quad a > 0, s > 0 \)
- \( \mathcal{L}\{f'(t)\} = sF(s) - f(0) \)
- \( \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \)