Notation: Throughout the exam $\Delta$ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the half-disc $D = \{z \in \mathbb{C} : |z| < 1, \text{Re} \, z > 0\}$ onto $\Delta$.

2. Let $D$ be a domain in $\mathbb{C}$ containing $0$ and $f : D \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 0$ and
\[
\int_{\partial R} f(z) \, dz = 0
\]
for every closed rectangle $R \subset D$ with sides parallel to the coordinate axes. Prove that $f(z) = 0$ for every $z \in D$.

3. Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in D$, and $f : D \rightarrow D$ be a holomorphic function such that $f(z_0) = z_0$. Show that $|f'(z_0)| \leq 1$.

4. Let $f_n : \Delta \rightarrow \Delta$, $n \geq 1$, be a sequence of holomorphic functions such that $f_n$ has a zero of order $m_n$ at $0$, where $\lim_{n \to \infty} m_n = \infty$. Show that $\{f_n\}$ converges locally uniformly to zero on $\Delta$. 