Instructions: Do as many problems as possible in the time allotted. In what follows, \( R \) is an associative ring with unity, all \( R \)-modules are unitary left modules, and \( 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \) is an exact sequence of \( R \)-modules.

1. We say that a homomorphism \( h : A \rightarrow M \) of \( R \)-modules can be extended to \( B \) if there exists a homomorphism \( \hat{h} : B \rightarrow M \) satisfying \( h = \hat{h}f \) (draw a diagram).

   For an \( R \)-module \( M \) prove that the following statements are logically equivalent. You may use a long exact sequence for the functor \( \text{Ext} \).
   (a) Every homomorphism \( A \rightarrow M \) can be extended to \( B \).
   (b) The sequence \( 0 \rightarrow \text{Hom}(C, M) \xrightarrow{\text{Hom}(g, M)} \text{Hom}(B, M) \xrightarrow{\text{Hom}(f, M)} \text{Hom}(A, M) \rightarrow 0 \) of abelian groups is exact.
   (c) The map \( \text{Ext}^1_R(C, M) \xrightarrow{\text{Ext}^1_R(g, M)} \text{Ext}^1_R(B, M) \) is a monomorphism of abelian groups.

   Recall that a monomorphism \( u : L \rightarrow M \) of \( R \)-modules is essential if, for all homomorphisms \( v : M \rightarrow N \) of \( R \)-modules, \( vu \) is a monomorphism if and only if \( v \) is a monomorphism. You may use the fact that if \( L \) is a submodule of \( M \), then the inclusion \( L \rightarrow M \) is an essential monomorphism if and only if \( L \cap X \neq 0 \) for all nonzero submodules \( X \) of \( M \).

2. If \( X \neq 0 \) is a submodule of an indecomposable injective \( R \)-module \( I \), prove that the inclusion \( i : X \rightarrow I \) is an essential monomorphism. You may use the existence of an injective envelope of \( X \), i.e., of an essential monomorphism \( j : X \rightarrow J \) where \( J \) is an injective \( R \)-module. Hint: what can you say about a homomorphism \( k : J \rightarrow I \) satisfying \( i = kj \)?

3. Let \( I \) be an indecomposable injective \( R \)-module and let \( u \) and \( v \) be \( R \)-endomorphisms of \( I \) that are not automorphisms. Using Problem 2, prove that \( u + v \) is not an automorphism of \( I \).

4. For a positive integer \( m \), set \( \mathbb{Z}/m = \mathbb{Z}/\mathbb{Z} \) and consider the exact sequence of abelian groups \( 0 \rightarrow \mathbb{Z} \xrightarrow{u} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0 \) where \( u \) is multiplication by \( m \) and \( v \) is the natural projection. For a positive integer \( n \), denote by \( d \) the greatest common divisor of \( m \) and \( n \).
   (a) Compute the kernel of the homomorphism \( v \otimes 1_{\mathbb{Z}/m} : \mathbb{Z} \otimes \mathbb{Z}/m \rightarrow \mathbb{Z}/m \otimes \mathbb{Z}/m \) of abelian groups.
   (b) Prove that \( \mathbb{Z}/m \otimes \mathbb{Z}/n \) is isomorphic to \( \mathbb{Z}/d \).

5. Suppose the ring \( R \) is left artinian and denote by \( J \) the Jacobson radical of \( R \).
   (a) Show that \( J^k/J^{k+1} \) is a semisimple module of finite length for all \( k \geq 0 \). (By definition \( J^0 = R \).)
   (b) Use the fact that \( J \) is nilpotent to show that \( R \) must be left Noetherian.