Solve the following 5 problems. Support your answers with sound reasons.

1. Let $k$ be a field, $k[x]$ the polynomial ring and $(x)$ the principal ideal of $k[x]$ generated by $x$. Standard properties of modules and quotient modules give us the following short exact sequence of $k[x]$ modules.

$$0 \rightarrow (x) \rightarrow k[x] \rightarrow k[x]/(x) \rightarrow 0$$

Prove whether or not this sequence splits.

2. Let $R$ be a ring with identity and let $P$ a left $R$-module.
   (a) State the definition given in terms of a lifting property of maps, of what it means for $P$ to be a projective module.
   (b) Prove that the definition given in (a) is equivalent to the following. The module $P$ is projective if and only if it is a summand of a free module.

3. Prove that if $R$ is a left artinian ring then $J(R)$ (the Jacobson radical of $R$) is a nilpotent ideal.

4. Let $m$ and $n$ be two not necessarily distinct integers both greater than or equal to 2. Consider the short exact sequence of $\mathbb{Z}$ modules

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

where the map $\mathbb{Z} \rightarrow \mathbb{Z}$ is multiplication by $m$. We may tensor over $\mathbb{Z}$ this sequence with $\mathbb{Z}/n\mathbb{Z}$ to obtain a new sequence.

$$(\ast) \quad 0 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

(a) For which pairs of integers $(m, n)$ is the sequence $(\ast)$ left exact? Prove it.
(b) For which pairs of integers $(m, n)$ is $(\ast)$ right exact? Prove it.

5. Let $R$ be a commutative ring with identity and $S \subset R$ a multiplicatively closed subset. There is an obvious natural homomorphism $f : R \rightarrow S^{-1}R$. State and prove a short easy to state condition on $S$ that is necessary and sufficient for $f$ to be injective.