1. Let $K$ be a field. A square matrix $A$ over $K$ is called unit upper triangular if it has 1s on the diagonal and 0s below the diagonal, that is, $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & i = j, \\ 0 & i > j. \end{cases}$$

The unipotent group $U_n(K)$ is the set of all $n \times n$ unit upper triangular matrices over $K$ with the usual matrix multiplication.

(a) (3 points) Prove that $U_n(K)$ is a subgroup of $\text{GL}_n(K)$.

(b) (7 points) If $K = \mathbb{F}_q$ is a field of order $q = p^e$, where $p$ is a prime number, show that $U_n(\mathbb{F}_q)$ is a Sylow $p$-subgroup of $\text{GL}_n(\mathbb{F}_q)$. (You will need to compute the order of $\text{GL}_n(\mathbb{F}_q)$.)

2. Let $G$ be a finite group with center $Z$. Let $p$ be a prime number. Prove the following statements.

(a) (5 points) If $|G| = p^e$ for some $e \geq 1$, then $Z$ is non-trivial.

(b) (5 points) If $G$ is non-Abelian and $|G| = p^e m$ for some $e \geq 1$ and some $m$ with $p > m$, then $G$ is not a simple group. (You will want to use the previous part.)

3. (10 points) Let $T$ be a self-adjoint linear operator on a finite-dimensional inner product space $V$, and assume that $T^n = \text{id}_V$ for some $n \geq 1$. Prove that $T^2 = \text{id}_V$. 
4. (a) (6 points) Let $V$ be a finite-dimensional real vector space with non-degenerate bilinear form $\langle - | - \rangle$. Let $T: V \to V$ be a linear operator. Prove that there exists an adjoint for $T$, that is, a linear operator $T^*: V \to V$ such that $\langle v | T^* w \rangle = \langle Tv | w \rangle$ for all $v, w \in V$. You may assume that $V$ has an orthonormal basis $\mathcal{B}$.

(b) (4 points) Let $V$ be the real vector space of all polynomial functions $f(t)$, with inner product $\langle f | g \rangle = \int_0^1 f(t)g(t) \, dt$. Let $D: V \to V$ be the derivative: $D(f) = f'$. Prove that there does not exist an adjoint $D^*$ for $D$. (Hint: consider $D^*(1)$.)

5. Let $R \neq 0$ be a commutative ring with identity.

(a) (6 points) Let $I$ be an ideal of $R$. Prove that $I$ is a free module if and only if it is a principal ideal generated by a non-zerodivisor.

(b) (4 points) Prove that if every finitely generated $R$-module is free, then $R$ is a field.

6. (10 points) Let $A$ be a $3 \times 3$ matrix with entries in $\mathbb{Q}$ such that $A^8 = I$. Prove that $A^4 = I$.

7. (10 points) Let $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}$. Let $E = \mathbb{Q}(\alpha, \beta)$. Prove that $[E : \mathbb{Q}] = 12$.

8. (10 points) Let $F$ be a field and let $f \in F[X]$ be an irreducible polynomial of degree $n$. Let $E$ be a splitting field of $f$. Prove that $[E : F] \leq n!$

9. (a) (6 points) Let $R$ be a commutative ring with identity. Assume that $\mathbb{Z}$ is a subring of $R$. You have seen that this makes $R$ into a $\mathbb{Z}$ module. Assume that $R$ is a finitely generated $\mathbb{Z}$ module. Prove that $R$ is not a field.

(b) (4 points) Find a field $F$ such that the additive group $(F, +)$ is a finitely generated $\mathbb{Z}$ module.