MAT 514 Final Exam

7 May 2010

This exam has 11 problems on 13 pages (including this cover page).
There is also a table of Laplace transforms.
You have two hours. You must show all your work; correct answers without complete justification will not receive full credit. Calculators are allowed.

Good luck!

Name: ____________________________________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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1. (8 pts) A very large tank initially holds 100 liters of pure, lifegiving water. Salty wa-
ter, with a salt concentration of 0.4 kilograms per liter, flows into the top of the tank at 5 liters/minute. The liquid in the tank drains through a hole in the bottom at 3 liters/minute. (Assume that the salt instantly mixes throughout the whole tank.) Find a differential equation for the amount of salt in the tank after \( t \) minutes (assuming the tank doesn’t overflow). Do not solve your equation.

Suppose the tank holds 300 liters. When does the tank first begin to overflow?
2. (10 pts) Solve the initial value problem and determine the largest interval in which the solution exists.

\[ ty' + 2y = 6t - t^2, \quad y(2) = 10 \]
3. **(a) (2 pts)** Show that the differential equation is exact.

\[(y^2 + 1) \, dx + (2xy + 1) \, dy = 0\]

**(b) (8 pts)** Find an equation implicitly defining solutions of the differential equation.

\[(y^2 + 1) \, dx + (2xy + 1) \, dy = 0\]
4. (10 pts) Find the solution of the initial value problem.

\[ 2y'' - 3y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]
5. (a) (4 pts) Find the general solution of the differential equation.

\[ y'' - 10y' + 25y = 0 \]
(b) (6 pts) Find the general solution of the differential equation.

\[ y'' - 10y' + 25y = 6e^{2t} \]
6. (10 pts) The function \( y_1(t) = t \) is a solution of the differential equation

\[
t^2 y'' - ty' + y = 0, \quad t > 0
\]

(you do not need to check this). Find the general solution.
7. (a) (8 pts) Let \( y \) be a solution of the initial value problem

\[
y'' + 3y' + 2y = 2 + t^3 + \delta(t - 3) + u_5(t)(2t - 10), \quad y(0) = 2, \quad y'(0) = -1.
\]

Find the Laplace transform \( Y(s) = \mathcal{L}(y) \). (Do not solve the initial value problem.)
(b) (8 pts) Find the inverse Laplace transform of the function.

\[ Y(s) = \frac{e^{-5s}}{s^2(s^2 + 4)} \]
8. (8 pts) Find the solution of the initial value problem

\[ y'' - 2y' - 3y = \delta(t - 4), \quad y(0) = 0, \quad y'(0) = 0. \]
9. (8 pts) Consider the following system of differential equations.

\[ x' = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} x \]

1. Find the general solution of the system.
2. Sketch the phase portrait, making sure to include all the typical types of trajectories.
10. (4 pts) Transform the following initial value problem into an initial value problem for two first order equations.

\[ u'' + 2u' + 3u = 4e^t \quad u(0) = 5 \quad u'(0) = 6 \]

11. (6 pts) Express \( f(t) \) in terms of the unit step functions \( u_c(t) \).

\[
\begin{aligned}
  f(t) &= \begin{cases} 
    t^2 & 0 \leq t < 2 \\
    6 - t & 2 \leq t < 4 \\
    2 & t \geq 4 
  \end{cases} 
\end{aligned}
\]