January 2014

Algebra Qualifying Examination
Homological Algebra Part

Solve 4 out of the following 6 problems:

1. Assume that $R$ is a ring such that $\text{pd}_R S \leq n$ for every simple $R$-module $S$. Show that $\text{pd}_R M \leq n$ for every module $M$ having a finite composition series. Here $\text{pd}$ denotes the projective dimension.

2. (a) Assume that $\xi: 0 \to A \to B \to C \to 0$ is a short exact sequence of $R$-modules, and that $A = A_1 \oplus I$ where $I$ is an injective module. Show that $\xi$ can be written as the direct sum of two short exact sequences, one of them being $0 \to I \to I \to 0 \to 0$ where the map $I \to I$ is an isomorphism.

   (b) Assume that $R$ is a ring where all the projective modules are injective. Prove that if $M$ is an $R$-module, then either $\text{pd}_R M = 0$ or $\text{pd}_R M = \infty$.

3. Assume $M_1 \subset M_2 \subset M_3 \subset \ldots$ is an ascending chain of submodules of a module $M$. Prove that $\varinjlim M_i = \bigcup_{i=1}^{\infty} M_i$.

4. A ring $R$ is called left hereditary if every submodule of a projective left module is again projective. For example, the ring of integers $\mathbb{Z}$ and the ring $K[t]$ of polynomials in one indeterminate over a field $K$ are both left (and right) hereditary.

   (a) Assume that $R$ is a left and right hereditary ring. Show that for every two $R$-modules $M$ and $N$ we have $\text{Ext}^i_R(M,N) = 0$ for each $i \geq 2$. Show also that for each right module $A$ and left module $B$ we have $\text{Tor}^i_R(A,B) = 0$ for all $i \geq 2$.

   (b) Give an example of a ring $R$ that is neither left nor right hereditary.

5. Let $\mathcal{T}$ be a triangulated category and let $u: A \to B$ be a monomorphism in $\mathcal{T}$. Prove that $u$ is an isomorphism from $A$ to a direct summand of $B$ (that is $u$ splits).

6. Let $\mathcal{A}$ be an abelian category and assume that the following diagram commutes and that the vertical arrows are isomorphisms.

   $\begin{array}{cccccc}
   0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
   \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
   0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 
   \end{array}$

Prove that the bottom row is exact if and only if the top row is exact.