August 2013

Algebra Qualifying Examination
Homological Algebra Part

Solve 4 out of the following 6 problems:

1. Let $K$ be a field and let $R = K[x]/\langle x^2 \rangle$. Set $x = x + \langle x^2 \rangle$ and let $S = R/Rx$ be a simple $R$-module. Find a projective resolution of $S$ where the $n$-th term $P^n = R$ for each $n \geq 0$. For each $n \geq 0$ compute the $K$-dimension of $\text{Ext}_R^n(S, S)$. What is the projective dimension of $S$?

2. Consider the following commutative diagram of $R$-modules where $R$ is a ring.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & 0 \\
0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & 0 \\
0 & \rightarrow & C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \rightarrow & 0 \\
0 & 0 & 0 & \\
\end{array}
\]

Assume that the columns are exact, and that the first two rows are also exact. Prove that the last row is also exact.

3. Let $\mathcal{T}$ be a triangulated category and let $L \xrightarrow{u} M \xrightarrow{v} N \rightarrow L[1]$ be a distinguished triangle. Prove that the composition $vu = 0$.

4. Let $R$ be a Noetherian ring and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of finitely generated $R$-modules. Assume that $\text{pd} A > \text{pd} B$. Prove that $\text{pd} C = \text{pd} A + 1$.

5. Let $(C, d)$ be a complex of $R$-modules and assume that there is a contracting homotopy $s$, that is a family of maps $s_n : C_n \rightarrow C_{n+1}$ satisfying

\[d_{n+1}s_n + s_{n-1}d_n = 1_{C_n}.\]
Prove that the complex \((\mathbb{C}, d)\) is exact.

6. Let \(R\) be a commutative ring and let \(L, M, N\) be three finitely generated \(R\)-modules. Prove that there is a natural isomorphism

\[
\tau_{L,M,N}: \text{Hom}_R(L \otimes_R M, N) \cong \text{Hom}_R(L, \text{Hom}_R(M, N))
\]