Instructions. There are 8 questions worth 200 points. Justify your answers with the necessary proofs, unless otherwise noted. While you should attempt every problem, try your best to answer questions fully. (Therefore it might be best to do first the problems you know how to do). Unless otherwise stated, assume that $\mathbb{R}$ and $\mathbb{C} = \mathbb{R}^2$ are equipped with their standard topology/metric.

1. (25 points) Let $L \subset S^3$ be a 2-component link (i.e., a subspace of $S^3$ consisting of two connected components $L_0, L_1 \subset S^3$, each of which is homeomorphic to $S^1$). Show that there is a continuous injection $f : S^3 \to S^3 \times [0, 1]$ such that $f(L_0) \subset S^3 \times \{0\}$ and $f(L_1) \subset S^3 \times \{1\}.$

2. (25 points) Let $S$ be the Sorgenfrey plane (i.e., the plane $\mathbb{R}^2$ equipped with the topology generated by the basis consisting of all rectangles of the form $[x, x + \delta) \times [y, y + \epsilon)$, for all $x, y \in \mathbb{R}$ and all $\delta, \epsilon > 0$). Determine which of the following are true:
   (a) $S$ is first countable.
   (b) $S$ is Lindelöf.
   (c) $S$ is connected.
   (d) $S$ is regular.
   (e) $\mathbb{Q}^2 \subset S$ equipped with the subspace topology is metrizable.

3. (25 points) Let $C(\mathbb{R}, \mathbb{R})$ denote the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$.
   (a) How does the topology of pointwise convergence on $C(\mathbb{R}, \mathbb{R})$ compare to the uniform topology (coarser/finer)?
   (b) Show that the set $\mathcal{B}(\mathbb{R}, \mathbb{R}) \subset C(\mathbb{R}, \mathbb{R})$ of bounded functions $f : \mathbb{R} \to \mathbb{R}$ is closed and open in the uniform topology.
   (c) Show that $C(\mathbb{R}, \mathbb{R})$ has uncountably many connected components with respect to the uniform topology. (Hint: Use part (b) to show that, for $r \neq s$, the functions $f_r(x) := rx$ and $f_s(x) := sx$ belong to different connected components).

4. (25 points) Let $S^1 := \{u \in \mathbb{C} \mid |u| = 1\}$ and $S^3 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}.$
   (a) State the Urysohn Lemma.
   (b) Let $L \subset S^3$ be a 2-component link (i.e., a subspace of $S^3$ consisting of two connected components $L_0, L_1 \subset S^3$, each of which is homeomorphic to $S^1$). Show that there is a continuous injection $f : S^3 \to S^3 \times [0, 1]$ such that $f(L_0) \subset S^3 \times \{0\}$ and $f(L_1) \subset S^3 \times \{1\}.$
   (c) Let $K \subset S^3$ be the knot $K := \{(z, w) \in S^3 \mid z^2 + \sqrt{2}w^3 = 0\}.$ Show that the map $f : S^1 \to S^3 - K$ given by $f(u) := (u, 0)$ is not nullhomotopic. (Hint: Compose $f$ with the map $g : S^3 - K \to S^1$ given by $g(z, w) := (z^2 + \sqrt{2}w^3)/|z^2 + \sqrt{2}w^3|).$

5. (25 points) Let $X$ be the space obtained by attaching an 8-cell to $\mathbb{R}P^7$ where the composition of the quotient map and attaching map $\partial e^8 \cong S^7 \to \mathbb{R}P^7 \to \mathbb{C}^7/\partial e^7 \cong S^7$ has degree 17. Since $\mathbb{R}P^3$ is a subspace of $\mathbb{R}P^7$, we have $\mathbb{R}P^3$ is a subspace of $X$.
   (a) Compute the homology groups of $X/\mathbb{R}P^3$ with $\mathbb{Z}$-coefficients.
   (b) Compute the homology groups of $X/\mathbb{R}P^3$ with $\mathbb{Z}_2$-coefficients.
6. (25 points) Recall the cone $CX$ of a space $X$ is the quotient of $X \times I$ by the relation $x \times 0 \sim x' \times 0$ for all $x, x' \in X$. Given a map $f : X \to Y$, the mapping cone $Cf$ is obtained by identifying $CX$ to $Y$ according to $x \times 1 \sim f(x)$ for all $x \in X$.

Let $f : S^1 \vee S^1 \to \Sigma_2$ be the injection pictured below:

(a) Show that $CX$ (for any space $X$) is contractible.
(b) Calculate $\pi_1(Cf)$ with basepoint at the image of the wedge point. Carefully describe generators and relators.
(c) Calculate $H_*(Cf)$. Carefully describe generators.

7. (25 points)
(a) Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.
(b) Let $(M, \partial M)$ be the Möbius band rel boundary. Show the transversal arc (the zigzag in the figure below) generates $H_1(M, \partial M)$ (hint: give $M$ a CW structure and use the definition of relative homology).

(c) Calculate $H_1(\mathbb{R}, \mathbb{Q})$ and find a basis (hint: recall that $\mathbb{Q}$’s path components are precisely the singletons).

8. (25 points) Consider a group presentation $G = \langle g_\alpha | r_\beta \rangle$. We describe a graph $\tilde{X}_G$. Let the vertices of $\tilde{X}_G$ be the elements of $G$, and at each vertex $g$, put an edge to $g \cdot g_\alpha$ for each generator $g_\alpha$ of $G$. We call $\tilde{X}_G$ the Caley graph of $G$ with respect to the generators $g_\alpha$. Each edge inherits the orientation $g \mapsto g \cdot g_\alpha$ from the generators $g_\alpha$, so some might call the Caley graph a directed graph.

(a) Show that the Caley graph of $G$ is connected.
(b) Fix a vertex $g$ in the Caley graph. Show that each relator $r_\beta$ determines a loop in the Caley graph based at $g$. (In fact, relators and their consequences are the only way to produce loops in the Caley graph).
(c) Draw the Caley graph of $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b | a^2, b^2 \rangle$.
(d) The group $G$ acts on its Caley graph $\tilde{X}_G$ by multiplying on the left: an element $g \in G$ sends a vertex $g' \in G$ to the vertex $gg' \in G$, and sends the edge from $g'$ to $g'g_\alpha$ to the edge from $gg'$ to $gg'g_\alpha$. This action is a covering space action (no justification required). Show that the regular covering spaces of $S^1 \vee S^1$ are precisely the Caley graphs of (presentations of) groups with two generators.