Instructions: Do all questions, justify your answers with the necessary proofs. All rings are associative (not necessarily commutative) with identity and all modules are left unitary modules. We denote by $\mathbb{Q}, \mathbb{R}$ the fields of rational and real numbers, respectively.

1. Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be an exact sequence of $R$-modules; such a sequence is called a short exact sequence. Prove the following statements. You may use the First Isomorphism Theorem.
   (a) (5 points) For any homomorphism $u : X \to B$ of $R$-modules satisfying $gu = 0$, there exists a unique homomorphism $v : X \to A$ satisfying $u = fv$.
   (b) (5 points) For any homomorphism $w : B \to Y$ satisfying $wf = 0$, there exists a unique homomorphism $v : C \to Y$ satisfying $w = vg$.

2. (15 points) Prove that the following statements are logically equivalent for the short exact sequence of Problem 1.
   (a) There exists a homomorphism $s : B \to A$ of $R$-modules satisfying $1_A = sf$.
   (b) There exists a homomorphism $t : C \to B$ of $R$-modules satisfying $1_C = gt$.
   (c) There exist homomorphisms $s : B \to A$ and $t : C \to B$ of $R$-modules satisfying $1_A = sf$, $1_C = gt$, and $1_B = fs + tg$.

3. The short exact sequence of Problem 1 is called a split short exact sequence if it satisfies any of the equivalent conditions of Problem 2.
   (a) (5 points) Give an example (with proof) of a short exact sequence that is not split.
   (b) (5 points) Give an example (with proof) of a split short exact sequence.

4. In this problem you may use that every module is a homomorphic image of a projective module and a submodule of an injective module; that a direct summand of a projective (resp., injective) module is projective (resp., injective); and that functor $\text{Hom}$ is left exact.
   (a) (15 points) Prove that the following two statements are logically equivalent for an $R$-module $C$.
      (i) For each exact sequence $0 \to L \overset{s}{\to} M \overset{t}{\to} N \to 0$ of $R$-modules, the sequence of abelian groups
          \[ 0 \to \text{Hom}_R(C, L) \overset{\text{Hom}_R(C, s)}{\longrightarrow} \text{Hom}_R(C, M) \overset{\text{Hom}_R(C, t)}{\longrightarrow} \text{Hom}_R(C, N) \to 0 \]
          is exact.
      (ii) Every exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is split short exact.
4. (continued)

(b) (15 points) Prove that the following two statements are logically equivalent for an $R$-module $A$.

(i) For each exact sequence $0 \rightarrow L \xrightarrow{s} M \xrightarrow{t} N \rightarrow 0$ of $R$-modules, the sequence of abelian groups

$$0 \rightarrow \text{Hom}_R(N, A) \xrightarrow{\text{Hom}_R(t, A)} \text{Hom}_R(M, A) \xrightarrow{\text{Hom}_R(s, A)} \text{Hom}_R(L, A) \rightarrow 0$$

is exact.

(ii) Every exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split short exact.

5. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$ of all $2 \times 2$ matrices $A = (a_{ij})$ satisfying $a_{11}, a_{21} \in \mathbb{R}, a_{22} \in \mathbb{Q}$, and $a_{12} = 0$, with the usual operations of matrix addition and multiplication.

(a) (5 points) Find the center of $R$, $Z(R) = \{ z \in R \mid zr = rz \text{ for all } r \in R \}$.

(b) (5 points) Is the ring $R$ left artinian?

(c) (5 points) Is the ring $R$ left noetherian?

(d) (5 points) Is the ring $R$ right artinian?

(e) (5 points) Is the ring $R$ right noetherian?

(f) (5 points) Find the radical of $R$, $J(R)$, and describe the ring structure of $R/J(R)$ in terms of $\mathbb{Q}$ and $\mathbb{R}$.

(g) (5 points) Describe the nonisomorphic simple left $R$-modules by indicating their underlying abelian group and $R$-action.