Instructions. There are 4 questions worth the total of 100 points. Do all questions, and justify your answers with the necessary proofs.

1. (30 points) Recall that the lower limit topology on \( \mathbb{Q} \) is generated by the basis sets \([a, b) \cap \mathbb{Q}\) for all pairs of rational numbers \(a, b\) with \(a < b\).
   (a) Use the Urysohn metrization theorem to show that \( \mathbb{Q} \) with the lower limit topology is metrizable.
   (b) Let \( \{X_\alpha\} \) be a collection of topological spaces. State the definitions of the product topology and the box topology on \( \prod X_\alpha \).
   (c) Show that if \( \{X_\alpha\} \) is a countable collection of second countable spaces, then \( \prod X_\alpha \) with the product topology is second countable.
   (d) Show that a metric space containing a countable dense subset is second countable.

2. (20 points) Let \( f: X \rightarrow Y \) be continuous.
   (a) Suppose \( Y \) is Hausdorff. Show that the set \( \{ (x, x') \mid f(x) = f(x') \} \) is closed in \( X \times X \).
   (b) Suppose that \( f \) is a quotient map and that the subspace \( f^{-1}(y) \subset X \) is connected for every \( y \in Y \). Show that if \( Y \) is connected, then so is \( X \).

3. (25 points) Let \( \mathcal{C}(X, Y) \) be the set of all continuous functions from a space \( X \) to a metric space \( Y \), and let \( C \subset X \) be a compact subspace and \( U \subset Y \) an open subset.
   (a) Show that the set \( S(C, U) := \{ f \in \mathcal{C}(X, Y) \mid f(C) \subset U \} \) is open in \( \mathcal{C}(X, Y) \) in the topology of uniform convergence.
   (b) Let \( f_n: [0, 1] \rightarrow \mathbb{R} \) (\( \mathbb{R} \) = the real numbers with the usual metric) be defined by
      \[
      f_n(x) := \min\{|nx - 1|, 1\}
      \]
      Show that \( \{f_n\} \) converges pointwise but not uniformly to the constant function \( f(x) = 1 \).
   (c) Give an example to show that \( S(C, U) \) need not be open in \( \mathcal{C}(X, Y) \) in the topology of pointwise convergence.

4. (25 points) Let \( X, Y \) be spaces with basepoints \( x \in X, y \in Y \).
   (a) Show that \( \pi_1(X \times Y, x \times y) \) is isomorphic to \( \pi_1(X, x) \times \pi_1(Y, y) \).
   (b) Show that there is no retraction of \( S^1 \times B^2 \) onto \( S^1 \times S^1 \) where \( S^1 \) is the unit circle in \( \mathbb{R}^2 \) and \( B^2 \) the closed unit disk in \( \mathbb{R}^2 \).
   (c) Let \( M_n(\mathbb{C}) \) be the set of all complex \( n \times n \) matrices with the topology induced from the standard topology on \( \mathbb{C}^{n^2} \) via the bijection \( A = (a_{ij}) \mapsto (a_{11}, a_{12}, \ldots, a_{21}, \ldots, a_{nn}) \), and let \( GL_n(\mathbb{C}) \) be the subspace of all matrices satisfying \( \det(A) \neq 0 \). Show that \( GL_n(\mathbb{C}) \) is not simply connected.