Some general information

- $1_A$ is the indicator function of the set $A$.
- The exponential distribution with parameter $\lambda > 0$ has
  
  $f(x) = \lambda \exp(-\lambda x)1_{[0,\infty)}(x)$, and
  
  characteristic function $\phi(t) = \lambda/(\lambda - it)$.

- If $X$ has the geometric distribution with parameter $p \in (0, 1)$ then
  
  $P(X = k) = p(1 - p)^{k-1}$, $k = 1, 2, \ldots$

- If $\Rightarrow$ denotes weak convergence, $X, X_n, Y_n$ are random variables, and $c > 0$ is constant, then $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ as $n \to \infty$ imply $X_n Y_n \Rightarrow cX$.

- In every problem you must prove each step, and refer to theorems by name whenever possible.

- You may use the conclusion of any previous problem, or previous problem part.
1. Let $\xi_1, \xi_2, \ldots$ be iid exponential random variables with parameter $\lambda$, and let $N$ be a geometric random variable with parameter $p \in (0, 1)$, independent of the $\xi_i$. Determine the distribution of

$$X = \xi_1 + \cdots + \xi_N$$

*Hint. Find the characteristic function of $X$.***
2. Let $A_1, \ldots, A_n$ be events and put $A = A_1 \cup \cdots \cup A_n$.

(a) Prove the following inequality for $n \geq 2$:

$$P(A) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

(b) Prove: if $n > 2$ and $P(A_i) > \frac{2}{n}$ for each $i$ then $P(A_i \cap A_j) > \frac{1}{n^2}$ for some $i \neq j$.

*Hint for (a). Prove the inequality: $1_A \geq \sum_{i=1}^{n} 1_{A_i} - \sum_{1 \leq i < j \leq n} 1_{A_i \cap A_j}$*
3. Let $X_1, X_2, \ldots$, be iid random variables. Prove that

$$\limsup_{n \to \infty} \frac{|X_n|}{\sqrt{n}} = \infty \quad a.s.$$ 

if and only if $E(X_1^2) = \infty$. 
4. Let $\xi_1, \xi_2, \ldots$ be iid random variables with finite mean $1$ and variance $3$. Let $S_n = \xi_1 + \cdots + \xi_n$ and $\bar{\xi}_n = S_n/n$. Prove that

$$Y_n = \frac{S_n - n}{\sqrt{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}}$$

converges in distribution as $n \to \infty$, and determine the limiting distribution.