1. Let $X$ be a metric space. Suppose that $A_n, n = 1, 2, 3, \ldots$ are nonempty compact subsets of $X$ such that $A_{n+2} \subset A_n \cup A_{n+1}$ for every $n \geq 1$. Prove that there exists a point $x \in X$ such that $x \in A_n$ for infinitely many values of $n$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to (0, \infty)$ are continuous functions. For $x \in \mathbb{R}$ define
   \[ h(x) = \sup_{0 < t < g(x)} f(t) \]
   (a) Prove that $h : \mathbb{R} \to \mathbb{R}$ is continuous.
   (b) Give an example in which $f$ is uniformly continuous on $\mathbb{R}$ but $h$ is not.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x + 1) = f'(x)$ for all $x \in \mathbb{R}$. Prove that the limit $\lim_{x \to +\infty} \frac{f(x)}{x}$ exists and is finite.

4. Let $f_n : \mathbb{R} \to \mathbb{R}, n = 1, 2, \ldots$, be $C^1$-functions; that is, continuously differentiable functions such that, for all $n$,
   \[ |f'_n(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1) \quad \text{and} \quad \int_0^1 f_n(x) \, dx = 0. \]
   Prove that the sequence $\{f_n\}$ has a subsequence that converges uniformly on $[0, 1]$.

5. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1$-mapping with $\det f'(x) > 0$ for all $x \in \mathbb{R}^2$. Assume that $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact. Prove that $f(\mathbb{R}^2) = \mathbb{R}^2$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$-function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $f$ takes the interval $[0, 1]$ onto itself. Prove that there is a sequence of polynomials $p_n : [0, 1] \to [0, 1]$ such that $p_n \to f$ uniformly on $[0, 1]$ and each $p_n$ is a strictly increasing function on $[0, 1]$. 

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