1. Show that a group of order 105 is not simple.

2. Let $G$ be a group with subgroups $H$ and $K$.
   (a) Let $x, y \in H$ with $x(H \cap K) = y(H \cap K)$. Prove that $xK = yK$.
   (b) Show that $[H : H \cap K] \leq [G : K]$, where $[G : K]$ denotes the index of $K$ in $G$.
   (c) If $[G : K]$ and $[G : H]$ are both finite, show that $[G : H \cap K]$ is finite.

3. (a) Let $G$ be a finite abelian group and assume that $m$ divides $|G|$. Show that $G$ has a subgroup of order $m$.
   (b) Give an example to show that the result in (a) is false if $G$ is not assumed to be abelian.

4. Let $A \in M_n(\mathbb{C})$ be a matrix over the complex numbers $\mathbb{C}$ with $A^* = -A$, where $A^*$ denotes the complex conjugate transpose of $A$. Let $\langle x, y \rangle = x^*y$ be the usual inner product on $\text{Col}_n(\mathbb{C})$.
   (a) Show that the eigenvalues of $A$ are purely imaginary.
   (b) If $\lambda$ and $\mu$ are distinct eigenvalues of $A$ with eigenvectors $v$ and $w$ in $\text{Col}_n(\mathbb{C})$ respectively, show that $\langle v, w \rangle = 0$.

5. Let $A \in M_n(\mathbb{C})$ be a matrix over the complex numbers $\mathbb{C}$.
   (a) If $A$ is similar to a diagonal matrix and $f(x) \in \mathbb{C}[x]$ is a polynomial, show that $f(A)$ is similar to a diagonal matrix.
   (b) If $A^2$ is similar to a diagonal matrix, does it follow that $A$ is similar to a diagonal matrix?

6. Let $i \in \mathbb{C}$ be the square root of $-1$.
   (a) Prove that $\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}[x]/(x^2 + 1)$.
   (b) Let $p \in \mathbb{Z}$ be a prime integer. Prove that $p$ is a prime element in $\mathbb{Z}[i]$ (a “Gaussian prime”) if and only if $x^2 + 1$ is an irreducible element of $\mathbb{F}_p[x]$. (Here $\mathbb{F}_p$ is the field with $p$ elements. You may use without proof the fact that $\mathbb{F}_p[x]$ is a PID.)

7. Let $\omega \in \mathbb{C}$ be a primitive 8th root of unity and set $F = \mathbb{Q}(\omega)$.
   (a) Prove that there are exactly three subfields $E \subset F$ with $[E : \mathbb{Q}] = 2$.
   (b) For each $E$ above, find (with justification) an element $a \in E$ such that $E = \mathbb{Q}(\alpha)$.

8. Let $R$ be a commutative ring and $M$ an $R$-module. An $R$-submodule $N$ of $M$ is called maximal if $N \neq M$ and there are no proper $R$-submodules of $M$ properly containing $N$.
   (a) Suppose $M$ is finitely generated. Prove that there exists at least one maximal $R$-submodule of $M$.
   (b) Prove that if $N$ is a maximal $R$-submodule of $M$, then $M/N \cong R/m$, where $m$ is a maximal ideal of $R$. 

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9. Reduce the matrix
\[
A = \begin{bmatrix}
3 & 1 & -4 \\
2 & -3 & 1 \\
-4 & 6 & -2
\end{bmatrix}
\]
to diagonal form over \( \mathbb{Z} \), and express the cokernel of \( A \) (that is, \( \text{Col}_3(\mathbb{Z})/\text{image}(A) \)) as a direct sum of cyclic groups.

10. Let \( F \) be a finite field. Prove that the multiplicative group \( F^\times \) of non-zero elements of \( F \) is a cyclic group. (Hint: a polynomial of degree \( n \) over a field has at most \( n \) roots.)