MAT 514

December 16, 2009

Professor Hsiang

FINAL EXAM

Show your work. No work-no credit!

Name ______________________ SU ID # ________________

I. Solve the given differential equations, i.e., if the initial value is given, solve the initial-value problem, otherwise, give the general solution.

(a) \( \frac{dy}{dt} = t^2 y^2 + y^2 \). [10 points]

(b) \( \frac{dy}{dt} = \frac{2t}{y + t^2 y} \), \( y(0) = -2 \) [10 points]

II. \( \frac{dy}{dt} = y^2(y^2 - 4) \).

(a) Sketch the phase line for the given differential equation. [10 points]

(b) Identify all the equilibrium solutions (critical points) as asymptotically stable solutions, unstable equilibrium solutions, or semistable equilibrium solutions. [10 points]

III. \( \frac{dy}{dt} = -2ty + e^{-t^2} \), \( y(0) = 2 \).

(a) For the given linear differential equation, find an integrating factor. [7 points]

(b) Find the general solution for the given differential equation. [7 points]

(c) Solve the initial-value problem. [6 points]

IV. For each of the following differential equations, determine if it is exact, find the solution for the exact equations, and find an integrating factor for the equations which are not exact, find an integrating factor only, but no need to solve the differential equations.

(a) \( (y \cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0 \) [10 points]

(b) \( (3xy + y^3)dx + (x^2 + xy)dy = 0 \) [10 points]

(continued on the other side.)
V. For each of the following given differential equations, (a) and (b),

(a) \( y'' + 5y' + 6y = e^t \), \( y(0) = 0, \ y'(0) = 1 \).
(b) \( y'' - 2y' + 2y = \sin 3t \), \( y(0) = 1, \ y'(0) = 0 \).

1. Write the corresponding homogeneous differential equation.  
   [2 points for each equation]
2. Find the characteristic polynomial of the corresponding homogeneous differential equation and its roots.  
   [2 points for each equation]
3. Using Wronskian to determine a set of linearly independent solutions for the corresponding homogeneous equation.  
   [2 points for each equation]
4. Using the method of underdetermined coefficients to find a particular solution of the given nonhomogeneous differential equation, and the general solution.  
   [2 points for each equation]
5. Solve the initial-value problem of the given differential equation.  
   [2 points for each equation]

VI. Using the method of Laplace transform to solve the following initial-value problems.

(a) \( y'' + 4y' + 3y = e^{2t} \), \( y(0) = 0, \ y'(0) = 1 \).  
   [10 points]
(b) \( y'' + 9y = \cos 2t \), \( y(0) = 1, \ y'(0) = 0 \).  
   [10 points]

VII. Solve each of the systems differential equations, i.e., if the initial value is given solve the initial value problem, otherwise give the general solution.

(a) \( \mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} \).  
   [10 points]
(b) \( \mathbf{x}' = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \).  
   [10 points]
(c) \( \mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x} \).  
   [10 points]