MAT 514 Final Exam

30 April 2009

This exam has 8 problems on 11 pages (including this cover page).
There is also a table of Laplace transforms.
You have two hours. You must show all your work; correct answers without complete justification will not receive full credit. Calculators are allowed.
Good luck!

Name: ____________________________

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1. (8 pts) A very large tank initially holds 100 liters of pure, lifegiving water. Salty water, with a salt concentration of 0.4 kilograms per liter, flows into the top of the tank at 5 liters/minute. The liquid in the tank drains through a hole in the bottom at 3 liters/minute. (Assume that the salt instantly mixes throughout the whole tank.) Find a differential equation for the amount of salt in the tank after \( t \) minutes (assuming the tank doesn't overflow). Do not solve your equation.
2. (12 pts) Solve the initial value problem.

\[ y' - 2y = 4 - t, \quad y(0) = -2 \]
3. (a) (4 pts) Show that the differential equation is exact.

\[(y^2 + 1) \, dx + (2xy + 1) \, dy = 0\]

(b) (8 pts) Find an equation implicitly defining solutions of the differential equation.

\[(y^2 + 1) \, dx + (2xy + 1) \, dy = 0\]
4. (12 pts) Find the solution of the initial value problem.

\[ 2y'' - 3y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]
5. (a) (6 pts) Find the general solution of the differential equation.

\[ y'' - 10y' + 25y = 0 \]
(b) (8 pts) Find the general solution of the differential equation.

\[ y'' - 10y' + 25y = 6e^{2t} \]
6. (12 pts) The function $y_1(t) = t$ is a solution of the differential equation

$$t^2 y'' - ty' + y = 0, \quad t > 0$$

(you do not need to check this). Find the general solution.
7. (a) (8 pts) Let $f(t)$ be a solution of the initial value problem

$$y'' + 3y' + 2y = 2 + \delta(t - 3) + \delta(t - 4), \quad y(0) = 2, \quad y'(0) = -1.$$ 

Find the Laplace transform $F(s) = \mathcal{L}(f(t))$. (Do not solve the initial value problem.)
(b) (10 pts) Find the inverse Laplace transform of the function.

\[ F(s) = \frac{e^{-5s} - e^{-10s}}{s^2(s^2 + 4)} \]
8. (12 pts) Find the solution of the initial value problem

$$y'' - 2y' - 3y = u_4(t), \quad y(0) = 0, \quad y'(0) = -1.$$