(1) In Euclidean space $\mathbb{R}^n$ with Lebesgue measure $m$, for $k \in \mathbb{N}$ and some $1 < p < \infty$ let $f, f_k \in L^p$ with $f_k \to f$ pointwise a.e. as $k \to \infty$. Assume that $\|f_k\|_p \leq M < \infty$ for all $k \in \mathbb{N}$. Also, let $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Prove or provide a counterexample to the statement: $\|f\|_p \leq M$.

(b) True or False, explain your answer. For all $R > 0$, for all $\delta > 0$ there is $F \subset \{x \in \mathbb{R}^n \mid |x| < R\} = B(0, R)$ with $m(F) < \delta$ and $f_k \to f$ uniformly on $B(0, R) \setminus F$.

(c) Prove or provide a counterexample to the statement: For all $\epsilon > 0$ there is a $R_0 > 0$ so that
$$\left( \int_{|x| \geq R} |g|^q \, dm \right)^{1/q} < \epsilon \text{ whenever } R > R_0.$$ 

(d) True or False, explain your answer. For all $\epsilon > 0$ there is a $\delta > 0$ so that for all $E \subset \mathbb{R}^n$ if $m(E) < \delta$ then $\int_E |g|^q \, dm < \epsilon$.

(e) Prove $\lim_{k \to \infty} \int f_k g \, dm = \int f g \, dm$.

(2) Let $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure as $n \to \infty$. Prove $f_n \to f$ in $L^1$ as $n \to \infty$.

(3) (a) Give an example of continuous $f : \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$ with $m(E) = 0$ so that $m(f(E)) \neq 0$, $m$ is Lebesgue measure on $\mathbb{R}$.

(b) Let $f$ be an absolutely continuous function on the interval $[a, b]$. Show that $m(f(E)) = 0$ for all $E \subset [a, b]$ with $m(E) = 0$.

(4) For $f$ a positive measurable function on the interval $[0, 1]$, which is larger (assume all the integrals make sense)?
$$\int_0^1 f \, dm \int_0^1 \log f \, dm \text{ OR } \int_0^1 f \log f \, dm$$

Prove your answer.