Put all work you wish to have graded in your blue book. No books, notes or
calculators are allowed. Do all parts of all questions. There are ten questions
each worth 10 points. Show the work you do to obtain an answer. Give reasons
for your answers. All the rings are commutative with identity unless otherwise
stated.

We use the standard notation: \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) are the usual rings of
integers, rational numbers, real numbers, and complex numbers.

1. Let \( G \) be a group and let \( H, K \) be two normal subgroups of \( G \) with
\( H \cap K = 1 \). Prove that \( HK \cong H \times K \).

2. Let \( G \) be a group of order \( p^2q \) where \( p, q \) are prime. Prove that \( G \) is not
simple.

3. Let \( G \) be a group of order 15 acting on a set of order 22. Assume there
are no fixed points. Determine how many orbits there are.

4. Let \( T : V \to V \) be a linear operator and let \( \{ v_1, \cdots, v_n \} \) be eigenvectors
with distinct eigenvalues. Prove that if \( a_1v_1 + \cdots + a_nv_n \) is an eigenvector, then
\textit{exactly one} of the coefficients is non-zero.

5. Let \( W \) be a subspace of a Euclidean space \( V \). (A Euclidean space is a
finite dimensional real inner product space.) Prove that \( W = W^\perp \perp \).

6. Let \( F \) be a finite field.
   (a) (6 points) Prove that the polynomial ring \( F[x] \) contains infinitely many
irreducible elements.
   (b) (2 points) Deduce from (a) that \( F[x] \) contains an irreducible element of
degree greater than 1.
   (c) (2 points) Deduce from (b) that \( F \) is not algebraically closed, hence any
algebraically closed field is infinite.

7. Let \( \mathbb{Q} \subset F \) be a field extension. Assume it is a Galois extension with
Galois group isomorphic to the symmetric group \( S_3 \). Prove that \( F \) is the
splitting field over \( \mathbb{Q} \) for an irreducible cubic polynomial \( f(x) \in \mathbb{Q}[x] \).
8. Let \( A \) be an \( 18 \times 18 \) matrix over \( \mathbb{C} \) with characteristic polynomial equal to \((x - 1)^6(x - 2)^6(x - 3)^6\) and minimal polynomial equal to \((x - 1)^4(x - 2)^4(x - 3)^3\). Assume \((A - I)\) has nullity 2, \((A - 2I)\) has nullity 3, and \((A - 3I)^2\) has nullity 4. Find the Jordan canonical form of \( A \).

9. Let \( R \) be a Noetherian integral domain with the property that any ideal that can be generated by 2 elements can actually be generated by 1 element. Prove that \( R \) is a principal ideal domain.

10. (a) (7 points) Let \( R \) be a commutative ring with identity. Assume that \( \mathbb{Z} \) is a subring of \( R \). You have seen that this makes \( R \) into a \( \mathbb{Z} \) module. Assume that \( R \) is a finitely generated \( \mathbb{Z} \) module. Prove that \( R \) is not a field.
    
    (b) (3 points) Find a field \( F \) such that the additive group \((F, +)\) is a finitely generated \( \mathbb{Z} \) module.