Algebra Preliminary Examination, January 13, 2011

Print name:

Solve as many problems as you can. Show your work, give reasons for your answers, provide all necessary proofs and counterexamples. There are 8 problems on 15 pages worth the total of 100 points. Check that you have a complete exam.

Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6

Problem 7

Problem 8

Total

Do NOT write mathematics on this page.
1. Let $P$ be the real vector space of polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $\leq n$, and let $D$ denote the derivative $\frac{d}{dx}$ considered as a linear operator on $P$.

(a) (6 points) Find the matrix of $D$ with respect to a convenient basis, and prove that $D$ is a nilpotent operator.
1. (continued)

(b) (6 points) Determine all the $D$-invariant subspaces. *Hint:* consider a polynomial of the highest degree in a $D$-invariant subspace.
2. Let $G$ be a group with a subgroup $H$ ($H$ need not be normal). The set $G/H$ of left cosets of $H$ in $G$ is a left $G$-set by means of $g \circ xH = gxH$, $g, x \in G$.

(a) (5 points) Prove that for each $a \in G$, the $G$-sets $G/H$ and $G/aHa^{-1}$ are isomorphic. Recall that a map $\phi : X \rightarrow Y$ of left $G$-sets is a homomorphism if $\phi(gx) = g\phi(x)$ for all $g \in G$, $x \in X$; an isomorphism is a bijective homomorphism; and $X, Y$ are isomorphic if there exists an isomorphism $X \rightarrow Y$. Hint: the right multiplication by $a^{-1}$ is a bijective map $G \rightarrow G$.

(b) (6 points) Let $K$ a subgroup of $G$. Prove that if the $G$-sets $G/H$ and $G/K$ are isomorphic, then $K = aHa^{-1}$, for some $a \in G$. Hint: if $\phi : X \rightarrow Y$ is an isomorphism of $G$-sets, compare the stabilizers of $x \in X$ and $\phi(x) \in Y$. 

2. (continued)

(c) (2 points) State the necessary and sufficient condition for the $G$-sets $G/H$ and $G/K$ to be isomorphic.
3. (a) (6 points) Prove that no group of order 56 is simple.
3. (continued)

(b) (7 points) Prove that a group of order 77 is cyclic.
4. (12 points) Let $A$ be the matrix of a real symmetric bilinear form $\langle , \rangle$ with respect to some basis. Prove or disprove: The eigenvalues of $A$ are independent of the basis.
5. Let $R$ be a commutative ring and $I$ an ideal of $R$.

(a) (4 points) Let $I[X] \subseteq R[X]$ be the subset of the polynomial ring consisting of polynomials with coefficients in $I$. Prove that $I[X]$ is an ideal of $R[X]$.

(b) (8 points) The quotients $R[X]/I[X]$ and $R[X]/(I, X)$ are isomorphic to $(R/I)[X]$ and $R/I$, not necessarily in that order. Decide which is which and prove your answers.
5. (continued)
6. Let $A$ be a square matrix over the complex numbers. Assume that the minimal polynomial of $A$ is $x^2(x - 5)$ and the characteristic polynomial of $A$ is $x^5(x - 5)^2$.

(a) (6 points) Give all the possible rational canonical forms for $A$. 
6. (continued) (b) (6 points) Give all the possible Jordan canonical forms for $A$. 
7. (12 points) An Abelian group is generated by four elements \{a, b, c, d\}, subject to the relations \(a + 3b + 3c + 5d = 0\), \(a + b + c = 0\), \(2b + 2c + 2d = 0\), and \(3c = 0\). Express this group as a direct sum of cyclic groups.
8. (14 points) Let $p$ be a prime integer and set $f(x) = x^p - 2 \in \mathbb{Q}[x]$. Determine the splitting field of $f$ and the elements of its Galois group over $\mathbb{Q}$. (You do not need to classify the structure of the group up to isomorphism, just its elements.)
extra sheet