1. Find a polynomial $p(x)$ of degree $\leq 2$ that satisfies

$$p(x_0) = a, \quad p'(x_0) = b, \quad p'(x_1) = c,$$

where $a$, $b$, $c$ are given constants and $x_0$, $x_1$ are two different points.

2. Let $f(0)$, $f(h)$ and $f(2h)$ be the values of a real valued function $f$ at $x = 0$, $x = h$ and $x = 2h$.

   (a) Derive the coefficients $c_0$, $c_1$ and $c_2$ so that

   $$Df_h(x) = c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

   is as accurate an approximation to $f'(0)$ as possible.

   (b) Derive the leading term of a truncation error estimate for the formula you derived in (a).

3. Consider the task of approximating a function $f(x)$ by a linear combination of $N$ function $q_k(x)$, $k = 1, 2, \ldots, N$, e.g.

   $$f(x) \approx \sum_{k=1}^{N} c_k q_k(x) \quad \text{for} \quad x \in [0, 1]$$

   (a) Give the equations that determine the $c_k$’s so that $\|f(x) - \sum_{k=1}^{N} c_k q_k(x)\|_2$ is minimized. (The norm is taken over the interval [0, 1].)

   (b) How would you solve the resulting equations?

4. Consider the three point Legendre-Gauss integration formula

   $$\int_{-1}^{1} f(x) \, dx \approx \frac{5}{9} f \left( -\sqrt{\frac{3}{5}} \right) + \frac{8}{9} f (0) + \frac{5}{9} f \left( \sqrt{\frac{3}{5}} \right)$$

   with the corresponding error expression

   $$\int_{-1}^{1} f(x) \, dx = \frac{5}{9} f \left( -\sqrt{\frac{3}{5}} \right) + \frac{8}{9} f (0) + \frac{5}{9} f \left( \sqrt{\frac{3}{5}} \right) + \frac{1}{525} f^{(4)}(\xi) \quad (1)$$

   for some point $\xi$, $|\xi| < 1$.

   (a) Derive the nodes and weights for the three point Legendre-Gauss quadrature for functions defined over the interval $[a, b]$.

   (b) Derive an error expression similar to (1) for the formula you obtain in (a).

5. Consider the fixed point iteration

   $$x_{n+1} = F(x_n)$$
(a) Assume that the fixed point iteration converges, explicitly derive the conditions on $F$ that ensure a second order rate of convergence.

(b) Using your result from (a), derive the condition on $\phi(x)$ so that the iteration

$$x_{n+1} = x_n + \phi(x_n) f(x_n)$$

will have a second order rate of convergence to a root $\alpha$ of the problem $f(x) = 0$.

6. Assume the points $\{x_i\}$, for $i = 1, 2, \ldots, n + 1$, are distinct. Prove that the polynomial of degree $\leq n$ that interpolates the data $\{(x_i, y_i)\}$ is unique.