1. If $F_1$ and $F_2$ are closed subsets of $\mathbb{R}^1$ and $\text{dist}(F_1, F_2) = 0$ then $F_1 \cap F_2 \neq \emptyset$. Prove or give a counterexample.

2. Newton’s method for finding zeroes of a function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is based on the recursion formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 1.$$  

Show that if $f \in C^1$, $f(a) = 0$ and $f'(a) \neq 0$, then there exists a $\delta > 0$ such that if $|x_1 - a| < \delta$ then $x_n \to a$. (Suggestion: use the Mean Value Theorem.)

3. Let $f : [0, \infty) \to [0, \infty)$ and for $h > 0$ and $k \geq 1$ set

$$M_k(h) = \sup_{(k-1)h \leq x < kh} f(x), \quad m_k(h) = \inf_{(k-1)h \leq x < kh} f(x).$$

Let

$$U(h) = \sum_{k=1}^{\infty} M_k(h)h, \quad L(h) = \sum_{k=1}^{\infty} m_k(h)h.$$  

We say $f$ is directly Riemann integrable if $U(h) < \infty$ for all $h > 0$ and

$$\lim_{h \downarrow 0} (U(h) - L(h)) = 0.$$  

Recall $f$ is improperly Riemann integrable on $[0, \infty)$ if $f$ is Riemann integrable on $[0, a]$ for every $a > 0$, and

$$\lim_{a \to \infty} \int_0^a f(t) \, dt < \infty.$$  

(a) Show that if $f$ is continuous and nonincreasing, then $f$ is directly Riemann integrable whenever $f$ is improperly Riemann integrable on $[0, \infty)$.

(b) Give an example of a continuous function $f$ which is improperly Riemann integrable on $[0, \infty)$ but not directly Riemann integrable.

4. Suppose $f : [0, \infty) \to [0, \infty)$ is such that for any sequence $a_n$ of nonnegative terms we have

$$\sum_{n=1}^{\infty} a_n < \infty \implies \sum_{n=1}^{\infty} f(a_n) < \infty.$$  

Prove that

$$\limsup_{x \to 0^+} \frac{f(x)}{x} < \infty.$$
5. Let \( f \) be a continuous real valued function defined on the unit square and for each \( 0 \leq x \leq 1 \) let \( f_x \) be the function on the unit interval defined by \( f_x(y) = f(x, y) \). Prove that for any sequence \( x_n \) in \([0,1]\) there is a subsequence \( n_k \) such that \( f_{x_{n_k}} \) converges uniformly on \([0,1]\).

6. If \( c \) is a real parameter prove that \( x^7 + x + c = 0 \) has a unique real root and that this root is a differentiable function of \( c \).