1. Let $X$ be a connected metric space. Given two points $p, q \in X$ and a number $\epsilon > 0$, prove that there exist an integer $n \geq 0$ and points $a_0, a_1, \ldots, a_n \in X$ such that $a_0 = p$, $a_n = q$, and $d(a_j, a_{j-1}) < \epsilon$ for all $j = 1, 2, \ldots, n$.

2. Suppose that $f: (0, 1] \to \mathbb{R}$ is a bounded continuous function such that for every $t \in \mathbb{R}$ the set $\{x \in (0, 1]: f(x) = t\}$ is finite. Prove that $f$ is uniformly continuous on $(0, 1]$.

3. Prove or disprove the following: if a function $f: (-1, 1) \to \mathbb{R}$ is differentiable on $(-1, 1)$ and $f'(0) = 0$, then for every $\delta > 0$ there exists $\epsilon > 0$ such that $\left| \frac{f(t) - f(s)}{t - s} \right| < \delta$ whenever $-\epsilon < s < t < \epsilon$.

4. Let $f$ be a bounded real-valued function on $[a, b]$ with a discontinuity at $c \in (a, b)$. Let $\alpha(x)$ be monotonically increasing on $[a, b]$ with $\alpha(c-) < \alpha(c) < \alpha(c+)$. Prove that $f$ is not Riemann-Stieltjes integrable with respect to $\alpha$ on $[a, b]$.

5. Give examples of sequences of functions $\{f_n\}$ and $\{g_n\}$ on $\mathbb{R}$ such that $\{f_n\}$ converges uniformly, $\{g_n\}$ converges uniformly but $\{f_ng_n\}$ does not converge uniformly on $\mathbb{R}$.

6. Let $\phi, \psi: \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable functions and define $F: \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (\phi(x, y, z), \psi(x, y, z), \phi^2(x, y, z) + \psi^2(x, y, z))$.

(a) Check whether or not the inverse function theorem applies to $F$ at any point $(x_0, y_0, z_0)$, i.e., check if $F$ satisfies the hypothesis of the inverse function theorem at any point $(x_0, y_0, z_0)$.

(b) Suppose that $F(\vec{a}) = \vec{b}$ for some points $\vec{a}, \vec{b} \in \mathbb{R}^3$. Explain geometrically why $F$ does not have an inverse function from an open set $V \subset \mathbb{R}^3$ containing $\vec{b}$ to an open set $U \subset \mathbb{R}^3$ containing $\vec{a}$. 
