Measure Theory Part

1. Let \( \{r_n\}_{n=1}^\infty \) be the rationals, \( f(x) = x^{-1/2} \) for \( 0 < x < 1 \) and 0 otherwise, and set \( g(x) = \sum_{n=1}^\infty 2^{-n} f(x - r_n) \). Is \( f(x) \) measurable? Why? Is \( g(x) \) measurable? Why? What is the set of points of discontinuity of \( g \)? Is \( g \) integrable? Why? Show that \( g \) is not in \( L^2 \) on any interval.

2. Let \( \mu \) be Lebesgue measure on the borel sets of the real line, and define \( \nu(E) \) to be 1 if \( 0 \in E \) and 0 if \( 0 \notin E \) for all borel sets \( E \). Is \( \nu \) a measure? \( \sigma \) finite? Compute \( \frac{d\nu}{d\mu} \).

3. Define \( L^p \) (Lebesgue measure). Is \( L^2(\mathbb{R}) \subseteq L^1(\mathbb{R}) \)? Why? Is \( L^2(0, 1) \subseteq L^1(0, 1) \)? Why?

4. Let \( f_k \to f \) in \( L^p \), \( 1 \leq p < \infty \), \( g_k \to g \) pointwise and \( \|g_k\|_\infty \leq M \) for all \( k \). Prove that \( f_k g_k \to fg \) in \( L^p \).

Complex Part

1. Let \( f \) be an analytic function on the unit disk and \( f(z) \) is real when \( z \) is real. Show that \( \overline{f}(z) = f(z) \).

2. Let \( \{f_n\} \) be a sequence of continuous functions on the closed unit disk that are analytic in the open unit disk. Suppose \( \{f_n\} \) converges uniformly on the unit circle. Show that \( \{f_n\} \) converges uniformly on the closed unit disk.

3. Suppose that \( f \) is an analytic function on an open set containing the closed unit disk, \( |f(z)| = 1 \) when \( |z| = 1 \) and \( f \) is not a constant. Prove that the image of \( f \) contains the closed unit disk.

4. Let \( \mathcal{F} \) be a family of analytic function

\[
    f(z) = z + \sum_{n=2}^\infty a_n z^n
\]
on the open unit disk such that \( |a_n| \leq n \) for each \( n \). Show that \( \mathcal{F} \) is normal, i.e. every sequence of functions in \( \mathcal{F} \) contains a subsequence converging normally to a function in \( \mathcal{F} \).