1. Let $X$ be a metric space and let $A_j$ be subsets of $X$, $j = 1, 2, \ldots$. For each of the following statements, prove it or give a counterexample (the $'$ means limit points):

(i) $(A_1 \cup A_2)' \subseteq A_1' \cup A_2'$

(ii) $\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} A_j$.

2. Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ is convergent and find its sum.

3. Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f''(0) \in \mathbb{R}$ exists. Prove that the limit $\lim_{x \to 0} \frac{f(2x) - 2f(x)}{x^2}$ exists.

4. (a) Let $f^4 \in \mathcal{R}$ (this means $f^4$ is integrable $dx$ on some closed interval) prove or disprove, $f \in \mathcal{R}$.
   (b) Let $f^5 \in \mathcal{R}$ prove or disprove, $f \in \mathcal{R}$.

5. Let $f(x, y)$ be a real continuous function on the rectangle $[0, 1] \times [0, 2]$. Given $\epsilon > 0$ show that there exists $n$ and real continuous functions $g_i(x)$ on $[0, 1]$ and $h_i(y)$ on $[0, 2]$ for $i = 1, \ldots, n$ so that

\[ |f(x, y) - \sum_{i=1}^{n} g_i(x)h_i(y)| < \epsilon \]

for all $(x, y)$ in the rectangle.

6. Given the equations $x - f(u, v) = 0$ and $y - g(u, v) = 0$ (a) give conditions that assure you can solve for $(x, y)$ in terms of $(u, v)$ and (b) similarly that you can solve for $(u, v)$ in terms of $(x, y)$. (c) Assuming these conditions are satisfied prove that

\[ \frac{\partial x(u, v)}{\partial u} \frac{\partial u(x, y)}{\partial x} = \frac{\partial y(u, v)}{\partial v} \frac{\partial v(x, y)}{\partial y} \]