MAT 286     Final Exam     May 5, 2003

Name(Print):.................................................................

Signature:.................................................................

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Instructions: Write the answers and show the main steps of your work on this test sheet. There are 14 questions on 14 pages (including this cover). Be sure you have all 14 pages (7 sheets) and that you do all 14 problems! The Final Exam is scored on a basis of 100 points and will count 20% of your final grade.

PUT YOUR NAME ON THE TOP OF EACH SHEET - NOW!
This Exam has 4 parts corresponding to the four sections of the course. You should spend no more than 25 minutes on each part to be sure that you get to the easier problems in each part. Where indicated, you must show your work to get full credit!

DO NOT WRITE ON THE REST OF THIS COVER SHEET!

Part I:

Part II:

Part III:

Part IV:

Total:

Have a nice and safe Summer!
Part I

Problem 1 (9 points) For each of the following questions, circle the correct answer.

a. Which of the following Riemann sums approximates \( \int_{1}^{3} \sin(x) \, dx \) using the left endpoints of 10 subintervals?

\[
\frac{1}{10} \left( \sin\left(1 + \frac{1}{10}\right) + \sin\left(1 + \frac{2}{10}\right) + \cdots + \sin\left(1 + \frac{10}{10}\right) \right);
\]

\[
\frac{1}{5} \left( \sin(1) + \sin\left(1 + \frac{1}{5}\right) + \cdots + \sin\left(1 + \frac{9}{5}\right) \right);
\]

\[
\frac{1}{5} \left( \sin(1) + \sin\left(1 + \frac{1}{5}\right) + \sin\left(1 + \frac{2}{5}\right) + \cdots + \sin\left(1 + \frac{10}{5}\right) \right);
\]

\[
\frac{1}{10} \left( \sin\left(1 + \frac{1}{10}\right) + \sin\left(1 + \frac{2}{10}\right) + \cdots + \sin\left(1 + \frac{20}{10}\right) \right);
\]

none of the above.

b. If \( f'(x) = x^6 + \frac{x^4}{3} - e^{-x} \) and \( f(0) = 5 \), then

\( f(x) = 6x^7 + \frac{4}{3}x^5 - e^{(x+1)}; \quad f(x) = \frac{x^7}{7} + \frac{x^5}{15} - e^x + 6; \)

\( f(x) = \frac{x^7}{7} + \frac{x^5}{15} + e^{-x} + 4; \quad \text{none of the above.} \)

c. The graph of the function \( f(x) \) is given below; circle the label of the graph of \( F(x) = \int_{0}^{x} f(t) \, dt \) from the next page:
Problem 2 (10 point)

a. (4 points) Make a rough sketch of the region between \( y = \frac{3x}{10} \) and \( y = \frac{8x - x^2}{10} \).

b. (2 point) Compute the \( x \) and \( y \) coordinates of the points of intersection. Show and label the intersection points on the above graph.

c. (2 point) Write down the integral that gives the area of this region.

d. (2 point) Compute the area of this region.
Problem 3 (6 points) A colony of 500 bacteria is placed in a dish at time \( t=0 \). It is observed that the rate of increase of the colony is given by:

\[
r(t) = \frac{5000t}{(t+2)^3} \text{ bacteria per day,}
\]

where time is measured in days.

a. (4 point) Compute the increase in the number of bacteria in the colony during the first \( n \) days since the colony was placed in the dish [INCLUDE APPROPRIATE UNITS!].

b. (2 point) In the long term, how many bacteria will be in the colony?
Part II

Problem 4 (12 points) For each of the following questions, circle the correct answer.

a. The daily sales of CD's in a record store are given, in dollars, by

\[ p(t) = \frac{3000t \cdot (365 - t)}{365} + \cos \left( \frac{2\pi t}{365} \right) \]

where time, t, is measured in days. The average daily sales over one year (365 days) is:

$192,500 \quad $18,250 \quad $1,182,500 \quad $182,500 \quad $318,259.

b. Is \( p(x) = \frac{x(6-x)}{18} \) a probability density function on the interval \( 0 \leq x \leq 6 \)?

yes \quad \text{no}

c. The mean of the probability density function \( p(x) = \frac{2x + 1}{6} \)

on \( 0 \leq x \leq 2 \) is:

\[
\begin{array}{cccc}
13 & 11 & 1 & \sqrt{13} - 1 \\
2 & 9 & 3 & 2 \\
\end{array}
\]

d. The median of the probability density function \( p(x) = \frac{2x + 1}{6} \)

on \( 0 \leq x \leq 2 \) is:

\[
\begin{array}{cccc}
13 & 11 & 1 & \sqrt{13} - 1 \\
2 & 9 & 3 & 2 \\
\end{array}
\]

Problem 5 (3 points) The random variable \( X \) has probability density function \( p(x) = \frac{x^2 \cdot (6-x)}{18} \)

on the interval \( 0 \leq x \leq 6 \). Which is more likely to occur (circle answer):

\( 0 \leq X \leq 3 \) or \( 3 \leq X \leq 6 \)?
Problem 6 (5 points) The length of rods manufactured at a certain plant can best be described by a normal density function. Suppose that the mean length of the rods is 72.05 inches and the standard deviation is 0.1 inches.

a. (3 points) A rod is randomly selected. What is the probability that the length of the rod is between 71.95 and 72.05 inches?

b. (2 points) A rod is randomly selected. What is the probability that the length of the rod is less than 71.95 inches?

Problem 7 (5 points) Consider a continuous stream of income at the constant rate of $3,500 per year deposited directly into a savings account accruing interest at an annual rate of 3.75%, compounded continuously. Find the future value, 45 years from now, of this savings account. Be sure to write down any integrals you calculate and formulas that you use.
Part III

Problem 8 (5 points) On the axes below, sketch the solution to the differential equation \( y' = \frac{2}{(y - 3)(y + 2)} \) that passes through the point \((0,0)\). The window is: \(-7.5 \leq x \leq 7.5, -2.5 \leq y \leq 4.5\).

Problem 9 (4 points) Find the solution in the form "\( y = \)" of the differential equation \( y' = \frac{3x^2 + 3}{2y} \) that satisfies the initial condition \( y(2) = -6 \).

Problem 10 (8 points) In part (a), circle the correct answer.

a. Which one of the slope fields on page 9 is the slope field of \( y' = \frac{x-1}{y-1} \)?

A B C D

b. On slope field A (not necessarily the solution to the previous problem) on page 9, sketch the solution through \((0,0)\).
All slope fields are drawn in the window 
$[-3.5 \leq x \leq 3.5 \text{ and } -1.5 \leq y \leq 1.5.]$
Problem 11 (8 points)

a. Nicotine leaves the body at a rate proportional to the amount of nicotine present, with constant of proportionality \(-0.34\).

(i) Let \(N\) be the level of nicotine in the body. Write a differential equation for \(N\).

(ii) Compute the general solution to this differential equation.

b. The velocity of a rocket is two-thirds of the square root of the distance \(D\) already traveled by the rocket.

(i) Give the differential equation for the distance \(D\) the rocket has traveled (recall that the velocity is the derivative of the distance with respect to time).

(ii) Compute the general solution to this differential equation. Give answer in the form \(D = \ldots\).
Part IV

Problem 12 (10 points) The following differential equations represent the interaction between two species:

\[
\frac{dx}{dt} = -3x + 4xy \quad \text{and} \quad \frac{dy}{dt} = -4y + xy.
\]

The sizes of the two populations (in thousands) are given by \(x\) and \(y\), respectively.

a. (3 points) Describe how the two species interact. Specifically, how would each species do in the absence of the other species? Are they helpful or harmful to each other? Explain/give reasons for answers.

b. (2 points) When \(x = 2\) and \(y = 1.5\),

- is \(x\) increasing decreasing (circle one);
- is \(y\) increasing decreasing (circle one).

c. (2 points) Give the differential equation relating the two populations:

\[
\frac{dy}{dx} =
\]
d. (3 points) On the coordinate system below, sketch the trajectory through the point (2,1.5). Label the point (2,1.5) on the trajectory and the direction. The window is $0 < x < 10$ and $0 < y < 5$. 
Problem 13 (9 points) Consider the SIR model of a flu epidemic given by:

\[ \frac{dS}{dt} = -0.003SI, \quad \frac{dI}{dt} = 0.003SI - 0.45I \quad \text{and} \quad S + I + R = 1000. \]

a. (2 points) \( I = I(t) \) represents the number of school boys who are infected at time \( t \) (measured in days) and \( S = S(t) \) represents the number of school boys who, at time \( t \), are susceptible to getting the flu but are not yet sick. What does \( R = R(t) \) represent?

b. (2 points) The differential equation relating \( I \) and \( S \) is:

\[ \frac{dI}{dS} = \]

c. (5 points) Let \( S_0 \) denote the threshold value for this model.

(i) (2 points) Compute \( S_0 \).

(ii) (3 points) What is the practical interpretation of the threshold value of this model?
Problem 14 (6 points) Yearly deposit of $2500 are made into a bank account that pays 6.5% interest per year, compounded annually. Each yearly deposit is made on the first day of the year and interest is added to the account on the last day of the year.

   a. (2 points) What is the balance in the account right after the 10th deposit?

   b. (2 points) What is the balance in the account right before the 10th deposit?

   c. (2 points) Right after the 10th deposit, how much of the balance comes from the annual deposits and how much from interest?