1. Let $A$ be the $4 \times 4$ real matrix

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
-2 & -2 & 2 & 1 \\
1 & 1 & -1 & 0
\end{bmatrix}.$$ 

Show that the characteristic polynomial for $A$ is $x^2(x - 1)^2$ and that it is also the minimal polynomial. Is $A$ similar over the field of complex numbers to a diagonal matrix?

2. Let $N_1$ and $N_2$ be $6 \times 6$ nilpotent matrices over the field $F$. Suppose that $N_1$ and $N_2$ have the same minimal polynomial and the same nullity. Prove that $N_1$ and $N_2$ are similar. Show that this is not true for $7 \times 7$ nilpotent matrices.

3. Let $V$ be an inner product space and $\beta$, $\gamma$ fixed vectors in $V$. Show that $T\alpha = (\alpha|\beta)\gamma$ defines a linear operator on $V$. Show that $T$ has an adjoint, and describe $T^*$ explicitly.
Now suppose $V$ is $\mathbb{C}^n$ with the standard inner product, $\beta = (y_1, \ldots, y_n)$, and $\gamma = (x_1, \ldots, x_n)$. What is the $j$, $k$ entry of the matrix of $T$ in the standard ordered basis? What is the rank of this matrix?

4. Let $V$ be the vector space of the polynomials over $\mathbb{R}$ of degree less than or equal to 3, with the inner product

$$(f|g) = \int_0^1 f(t)g(t)dt.$$ 

If $t$ is a real number, find the polynomial $g_t$ in $V$ such that $(f|g_t) = f(t)$ for all $F$ in $V$. Let $D$ be the differentiation operator on $V$. Find $D^*$.

5. Let $V$ be a finite-dimensional inner product space, and let $W$ be a subspace of $V$. Then $V = W \oplus W^\perp$, that is, each $\alpha$ in $V$ is uniquely expressible in the form $\alpha = \beta + \gamma$, with $\beta$ in $W$ and $\gamma$ in $W^\perp$. Define a linear operator $U$ by $U(\alpha) = \beta - \gamma$.

(a) Prove that $U$ is both self-adjoint and unitary.
(b) If $V$ is $\mathbb{R}^3$ with the standard inner product and $W$ is the subspace spanned by $(1, 0, 1)$, find the matrix of $U$ in the standard ordered basis.

6. Prove that a normal and nilpotent operator is the zero operator.