1. Let $f$ be a real valued function defined on a set $D$ which is dense in $[0,1]$. If $f$ is uniformly continuous on $D$, show that $f$ can be extended to a uniformly continuous function on $[0,1]$.

2. Fix a real number $a > 1$ and define a sequence of numbers $\{x_n\}$ inductively by $x_1 = 0$ and 

$$x_{n+1} = \frac{a(1+x_n)}{a+x_n} \quad \text{for } n = 0, 1, \ldots$$

Show that $\lim_{n \to \infty} x_n$ exists and find this limit.

3. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a given function, and consider 3 variables $x, y, z$ which are related by the equation

(A) \quad f(x, y, z) = 0.

In some textbooks in Thermodynamics it is claimed that (A) implies the formula

(B) \quad \left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1 ,

where it is understood that equation (A) may be solved for each variable in terms of the other two. Thus, for example, $z$ may be expressed as function of $y$ and $z$, and it is this function which is differentiated in the symbol $\frac{\partial x}{\partial y}$. By making use of the Implicit Function Theorem, formulate and prove a precise theorem, including appropriate hypotheses on $f$, which shows that (B) is indeed a consequence of (A).

4. Let $f$ be a continuous, nonnegative function defined on $[a, b]$ with $M = \sup_{x \in [a, b]} f(x)$. Prove 

$$\lim_{n \to \infty} \left( \int_a^b f^n(x) \, dx \right)^{1/n} = M .$$

5. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$, and $f(s,t) = t^3/(s^2 + t^2)$ otherwise. Prove the following facts:

(a) $f$ is continuous on $\mathbb{R}^2$,
(b) $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ exist at all points of $\mathbb{R}^2$,
(c) $f$ is not differentiable at $(0,0)$.