1. Let \( \{a_n\} \) be a sequence of nonnegative numbers such that \( \sum_{n=0}^{\infty} a_n = 1 \). The power series
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]
converges for all \( x \in [-1,1] \). If \( L \) denotes the left-hand derivative of \( f \) at \( x = 1 \), \( L = \lim_{x \to 1^-} \frac{f(1) - f(x)}{1-x} \), show that
\[
L = \sum_{n=1}^{\infty} na_n,
\]
including the case \( +\infty = +\infty \).

2. Let \( \{f_n\} \) be a sequence of continuously differentiable functions on \( \mathbb{R} \) such that \( f_n(0) = 0 \) for all \( n \), \( f'_n \cdot f'_m \equiv 0 \) for all \( n \neq m \), and \( f'_n \to 0 \) uniformly on \( \mathbb{R} \) as \( n \to \infty \).
   (a) Prove that \( \sum_{i=1}^{\infty} f'_n \) converges uniformly and absolutely on \( \mathbb{R} \). Let \( g = \sum_{i=1}^{\infty} f'_n \).
   (b) Prove that \( \sum_{i=1}^{\infty} f_n \) converges pointwise on \( \mathbb{R} \). Let \( f = \sum_{i=1}^{\infty} f_n \).
   (c) Show that \( f \) is differentiable on \( \mathbb{R} \), and that \( f'(x) = g(x) \) for all \( x \in \mathbb{R} \).

3. Let \( g, f_n, n = 1, 2, \ldots \) be real valued functions defined on \( [0, \infty) \) such that: (i) each \( f_n \) is Riemann integrable on every interval \( [0, T] \), \( T < \infty \); (ii) \( |f_n(x)| \leq g(x) \) for all \( n \) and \( x \); (iii) \( \int_0^\infty g(x)dx < \infty \), and (iv) there is a function \( f \) such that \( f_n \to f \) uniformly on every interval \( [0, T] \) as \( n \to \infty \). Prove, without using results from Lebesgue integration theory, that the improper Riemann integrals \( \int_0^\infty f_n(x)dx \) and \( \int_0^\infty f(x)dx \) exist, and
\[
\lim_{n \to \infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.
\]

4. Determine the convergence (absolute or conditional) or divergence of the following series:
   (a) \( \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}} \)
   (b) \( \sum_{n=1}^{\infty} n^2[\pi^{1/n} - 1]^n \)
   (c) \( \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)} \)
   (d) \( \sum_{n=1}^{\infty} n! e^{-n} \)