1. Let \( T: V \rightarrow W \) be a linear transformation of finite dimensional vector spaces. Assume that rank \( T = k \). Prove that there exist ordered bases \( B \) for \( V \), and \( C \) for \( W \), such that the matrix representation of \( T \) with respect to \( B \) and \( C \) has the following property: its \((i,i)\) entry equals one for \( i = 1,2,\ldots,k \), and all its other entries are zero.

2. Suppose \( V = W_1 \oplus W_2 \) and that \( f_1 \) and \( f_2 \) are inner products on \( W_1 \) and \( W_2 \) respectively. Show that there is a unique inner product \( f \) on \( V \) such that
   (a) \( W_2 = W_1^f \);
   (b) \( f(\alpha,\beta) = f_k(\alpha,\beta) \) when \( \alpha,\beta \) are in \( W_k \), \( k = 1,2 \).

3. Let \( V \) be an \( n \)-dimensional vector space and let \( T \) be a linear operator on \( V \). Suppose that there exists a positive integer \( k \) such that \( T^k = 0 \). Prove that \( T^n = 0 \). What is the characteristic polynomial for \( T \)?

4. Suppose \( B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & 2 \end{pmatrix} \). Find:
   (a) the characteristic polynomial and the eigenvalues of \( B \);
   (b) a maximal set \( S \) of linearly independent eigenvectors of \( B \).
   (c) Is \( B \) diagonalizable?

5. If \( A \) is a square matrix with characteristic polynomial \( f(x) = (x-2)^3 (x+3)^4 \) and minimal polynomial \( g(x) = (x-2)(x+3)^2 \), give all possible Jordan normal forms for \( A \).

6. Let \( T: V \rightarrow W \) be a linear transformation with \( \text{dim } V = n \), \( \text{dim } W = m \), and rank \( T = k \). Let \( T^*: W^* \rightarrow V^* \) be the dual linear transformation. What are the rank and the nullity of \( T^* \)?