Preliminary Exam, August 1995.

1. Suppose $A$ is a matrix over the complex numbers with characteristic polynomial $(x + 2)^2(x - 1)^5$. If the rank of $(A - I)^2$ is $3$ and the rank of $(A + 2I)$ is $5$, where $I$ denotes the identity matrix, what are the possibilities for the Jordan Canonical Form of $A$?

2. Suppose that $E$ is an idempotent linear operator on a vector space, that is $E^2 = E$. Show that the only possible characteristic values for $E$ are $0$ and $1$.

3. Suppose $V$ is a vector space with a finite spanning set $S = \{v_1, v_2, \ldots, v_n\}$. Show that $S$ contains a basis for $V$.

4. Assume $V$ is a finite dimensional vector space of dimension $n$ and let $T$ and $S$ be linear operators on $V$, both with rank strictly greater than $\frac{n}{2}$. Show that the composition $SoT$ is nonzero.

5. (a) Suppose $T:V \rightarrow W$ is a linear transformation between the vector spaces $V$ and $W$. What is meant by $T^t$, the transpose of $T$?

(b) Assume $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by $S(x,y) = (x+y, 2x-y)$. Let $\{f_1, f_2\}$ be the dual basis of the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$, where $e_1 = (1,0)$ and $e_2 = (0, 1)$. Find $S^t(f_2)$.

6. (a) Let $V$ be an inner product space with inner product $(\cdot, \cdot)$, and assume $T:V \rightarrow V$ is a linear operator on $V$. What does it mean to say that $T$ is self adjoint? What does it mean to say that $T$ is normal?

(b) Let $P_2$ be the inner product space of polynomials of degree at most two over the real numbers, with the inner product $(fg) = \int_1^1 fg$. If $\phi$ is a linear functional defined on $P_2$ by $\phi(f) = f(0)$, find $h \in P_2$ with $\phi(f) = (f|h)$. 