1. A 5-by-5 matrix $A$ has characteristic polynomial $(x - 2)^3(x + 1)^2$, while the matrix $(A - 2I_5)^2$ has rank 2 and $A + I_5$ has rank 4. What are the possible Jordan canonical forms of $A$?

2. If $A$ is a Hermitian complex matrix, show that its characteristic values must be real. [Recall that $A$ is called Hermitian (or self adjoint) if it satisfies the equation $A = A^\dagger$, where $A^\dagger$ is the complex conjugate of the transpose of $A$.]

3. Let $V$ be a vector space with basis $B = \{v_1, v_2, \ldots, v_n\}$ and let $w \in V$ be nonzero. Show directly, without quoting the dimension theorem, that we can find $i$ such that we can replace $v_i$ in $B$ by $w$ and still have a basis for $V$.

4. Let $V$, $(\cdot, \cdot)$ be a finite dimensional inner product space over the real numbers. If $W$ is a subspace of $V$, prove that we can write $V$ as a direct sum $V = W \oplus W^\perp$, where $W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$.

5. Let $V$ and $W$ be finite dimensional vector spaces over a field $k$ and let $T: V \to W$ be a linear transformation.

   (a) Define the transpose map $T^*: W^* \to V^*$ where $W^* = \text{Hom}_k(W, k)$ is the dual of $W$.

   (b) Show that $T^*$ is injective if and only if $T$ is surjective.

6. Let $V$ and $W$ be finite dimensional vector spaces over a field $k$ and let $T: V \to W$ be a linear transformation. Let $S = \{v_1, v_2, \ldots, v_m\}$ be a subset of $V$. For each of the following statements either prove it or give a counter example to it.

   (a) If $S$ is linearly independent set in $V$, then $\{T(v_1), T(v_2), \ldots, T(v_m)\}$ must be a linearly independent set in $W$.

   (b) If $\{T(v_1), T(v_2), \ldots, T(v_m)\}$ is a linearly independent set in $W$, then $S$ must be a linearly independent set in $V$. 