Algebra Preliminary Examination, August 21, 2006

Print name: ___________________________ Score: ___________

Show your work, give reasons for your answers, provide all necessary proofs and counterexamples. There are 10 problems on 17 pages worth 10 points each for the total of 100 points. Check that you have a complete exam, print your name on each page.

1. (a) (3 points) If $G$ is an abelian group, prove that the map $\phi : G \rightarrow G$ defined by $\phi(g) = g^m$, for all $g \in G$ and some integer $m > 0$, is an endomorphism.

(b) (2 points) Give an example showing that one cannot drop the assumption that $G$ is abelian in (a).
(c) (5 points) In the setting of (a), suppose that the order of $G$ is $n$ and that the integers $m$ and $n$ are coprime. Prove that the map $\phi : G \to G$ is an automorphism.
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2. (a) (2 points) If $G$ is a group, $S$ is a left $G$-set, and Perm $S$ is the group of permutations of $S$, the map $\Phi : G \to$ Perm $S$ defined by $\Phi(g)(s) = gs$, for all $s \in S$, is a homomorphism of groups. Using this fact, prove that $N = \{g \in G \mid gs = s$ for all $s \in S\}$ is a normal subgroup of $G$.

In the rest of the problem, let $H$ be a subgroup of $G$ and let $S$ be the set of left cosets of $H$ in $G$.

(b) (2 points) Prove that $N \subseteq H$. 
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2. (continued) Let $|G| = n < \infty$ and $[G : H] = k > 1$.
(c) (3 points) Prove that if $n > k!$ then $\{1\} \neq N \neq G$.

(d) (3 points) If $k$ is the least prime dividing $n$, prove that $H = N$. *Hint:* find the cardinality of $\text{Im} \Phi$. 
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3. (10 points) Prove that a group of order 35 is cyclic.
4. Let $V$ be a fixed vector space of finite dimension $n > 0$ over a field $F$, and let $\lambda \in F$. If $T : V \to V$ is a linear operator with an eigenvalue $\lambda$, let $m$ be the maximal number of linearly independent eigenvectors with eigenvalue $\lambda$.

(a) (5 points) Prove that the multiplicity of $\lambda$ as a root of the characteristic polynomial of $T$ is at least $m$. 

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4. (continued)
   (b) (5 points) Among all linear operators on $V$ with an eigenvalue $\lambda$, what are the smallest and largest possible values of $m$? Justify your answer (remember, $n$ is arbitrary but fixed).
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5. Let $V$ be a finite-dimensional complex vector space with a positive definite hermitian form $\langle \, , \rangle$, and let $T : V \rightarrow V$ be a linear operator.

(a) (2 points) Give the definition of the adjoint operator $T^* : V \rightarrow V$.

(b) (2 points) Give the definition of when $T$ is a normal linear operator.
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5. (continued)
(c) (6 points) Assuming $T$ is normal, prove that $\text{Ker } T = (\text{Im } T)^\perp$ where, for a subspace $W$ of $V$, $W^\perp$ denotes the orthogonal compliment of $W$. 
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6. Let $R$ be a commutative ring with identity and let $I \subseteq R$ be an ideal. Define the radical of $I$ denoted $\sqrt{I}$ by $\sqrt{I} = \{ r \in R : r^n \in I \text{ for some positive integer } n \}$.

(a) (4 points) Prove that $\sqrt{I}$ is an ideal of $R$. 
6. (continued)

(b) (6 points) We say that $I$ is a radical ideal if and only if $I = \sqrt{I}$. Recall that an element $r$ is called nilpotent if and only if $r^n = 0$ for some positive integer $n$. Prove that $I$ is a radical ideal if and only if $0 + I$ is the only nilpotent element of the quotient ring $R/I$. 
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7. Let $R$ be a PID and let $a$ and $b$ be two nonzero nonunits in $R$.
   a) (2 points) Give the definition of the greatest common divisor of $a$ and $b$.

   (b) (4 points) Prove that a greatest common divisor of $a$ and $b$ exists.
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7. (continued)
   (c) (4 points) Let $c$ be a greatest common divisor of $a$ and $b$. Prove there exist $x, y \in R$ such that $c = xa + yb$. 
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8. Let \( F \) be a finite field, \( F[x] \) the polynomial ring over \( F \), and \( M \) an \( F[x] \) module.

(a) (2 points) Explain why \( M \) is also an \( F \) vector space in a natural way. Denote by \( \dim_F(M) \) the dimension of \( M \) as an \( F \) vector space.

(b) (8 points) Prove that for each positive integer \( n \) there exists a simple \( F[x] \) module \( M_n \) such that \( n < \dim_F(M_n) < \infty \).
9. Let $A$ be a square matrix over the complex numbers. Assume the characteristic polynomial of $A$ is $(x - 2)^4(x - 3)^5$. Also assume that nullity $(A - 2I) = 4$ and nullity $(A - 3I) = 1$.

(a) (6 points) What are the possible Jordan normal forms of $A$?

(b) (4 points) For each possible Jordan normal form of $A$ give its minimal polynomial.
10. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial ($\mathbb{Q}$ is of course the rational numbers) and let $F$ be the splitting field for $f(x)$ over $\mathbb{Q}$.

(a) (4 points) Prove that the Galois group of $F$ over $\mathbb{Q}$ is isomorphic to either $S_3$ or $\mathbb{Z}/3\mathbb{Z}$. 

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10. (continued) For (b) and (c) suppose that \( f(x) = x^3 - x + 2 \).
(b) (3 points) Prove that \( f(x) \) is irreducible over \( \mathbb{Q} \).

(c) (3 points) Still denoting by \( F \) the splitting field for \( f(x) \) over \( \mathbb{Q} \) decide which of the two possibilities in (a) for the Galois group of \( F \) over \( \mathbb{Q} \) is the case for this \( f(x) \).