FINAL EXAM
MAT 514
December 18, 2001

NAME __________________________

Instructions. Write the answers and show your work on the test in the space provided. Calculators may not be used on the exam. This exam has 11 pages and a total of 120 points.

1. (10 pts) Consider the autonomous differential equation $\frac{dy}{dt} = 2y(y^2 - 4y + 3)$.

   (a) Find and classify all the equilibrium points and draw the phase line for the equation.

   (b) If $y(t)$ is the solution of the differential that satisfies the initial condition $y(0) = 2$, find $\lim_{t \to \infty} y(t)$.
2. (10 pts) Initially, a tank contains 100 gallons of water in which 10 pounds of salt are dissolved. A solution containing 2 pounds of salt per gallon of water is then pumped into the tank at the rate of 5 gallons per minute. The mixture is kept uniform by stirring and it flows out of the tank at the same rate of 5 gallons per minute. Find an expression for the amount of salt in the tank at any time $t$. 
3. (10 pts) Consider the one-parameter family of differential equations \( \frac{dy}{dt} = y^2 - 4y + \alpha \). Draw the bifurcation diagram and find all the bifurcation points.
4. (10 points) Find the general solution of the following linear system:

\[
\frac{dx}{dt} = x - y \\
\frac{dy}{dt} = x + y
\]
5. (10 points) The matrix for the linear system

\[
\begin{align*}
\frac{dx}{dt} &= x - 3y \\
\frac{dy}{dt} &= 3x - 5y
\end{align*}
\]

has only \( \lambda = -2 \) as an eigenvalue and the corresponding eigenvector is \( V = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Find the general solution of the system.
6. (10 pts) Solve the following second order initial value problem:

\[
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 8e^{-2t}, \quad y(0) = 1, \quad y'(0) = -2
\]
7. (10 pts) Consider the following non-linear system:

\[
\frac{dx}{dt} = y^2 - 2xy + 2x \\
\frac{dy}{dt} = x - y
\]

(a) Find all equilibrium points.

(b) Classify each of the equilibrium points.
8. (7 pts) Consider the following non-linear system:

\[ \frac{dx}{dt} = 2xy - y^2 \]
\[ \frac{dy}{dt} = x + y - 1. \]

Sketch the \( x \)-nullcline and include the direction vectors along the nullcline in the sketch.

9. (8 pts) Let the function \( v(t) \) be defined as follows:

\[ v(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 2, \\
0 & \text{if } t > 2.
\end{cases} \]

Find \( \mathcal{L}(v) \), the Laplace transform of \( v(t) \).
10. Consider the differential equation \( \frac{dy}{dt} = -y + 2u_3(t) \).

(a) (7 pts) Sketch, roughly, the slope field for this differential equation.

(b) (8 pts) Solve the initial value problem \( \frac{dy}{dt} = -y + 2u_3(t), \ y(0) = 1 \).
11. (10 pts) Use Laplace transforms to solve the initial value problem
\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = u_3(t), \quad y(0) = 1, \quad y'(0) = -1.
\]
12. (10 pts) Use Laplace transforms to solve the initial value problem

\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \delta_3(t), \quad y(0) = 1, \quad y'(0) = -1.
\]